Chapter 10

Plumes and Thermals

SUMMARY: This chapter describes several distinct structures that fluids develop in reaction to localized inputs of buoyancy. A punctual and sustained source of buoyancy usually creates a continuous rise of lighter fluid through the ambient denser fluid, with mixing occurring along the way. Such structure is called a *plume*. Should the process be intermittent, the rising buoyant fluid parcels are called *thermals*. *Buoyant jets* are plumes with the added propulsion of momentum, and *buoyant puffs* are fluid parcels that rise under the combined action of buoyancy and momentum.

10.1 Plumes

Plumes are common features in environmental fluids, which occur whenever a persistent source of buoyancy creates a rising motion of the buoyant fluid upward and away from the source. The clearest example is that of hydrothermal vents at the bottom of the ocean (Figure 10.1). Another occurrence is the rising of freshwater from the bottom of the sea at submarine springs in karstic regions such as along the Dalmatian Coast of Croatia, where such features are called *vrulje*. The common urban smokestack plume is, however, somewhat different because the warm gas rises not only under its own buoyancy but also under the propulsion of momentum (inertia). Such plume is more properly categorized as a buoyant jet or forced plume.

What drives a plume is its *heat flux*, defined as the amount of heat (expressed in joules) being discharged through the exit hole per unit time. Because it is more practical in later mathematical developments, this quantity is divided by $\rho_0 C_p$ (the fluid's reference density and heat capacity at constant pressure) and then multiplied by αg (the fluid's thermal expansion coefficient and the gravitational acceleration), giving rise to the *buoyancy flux* F:

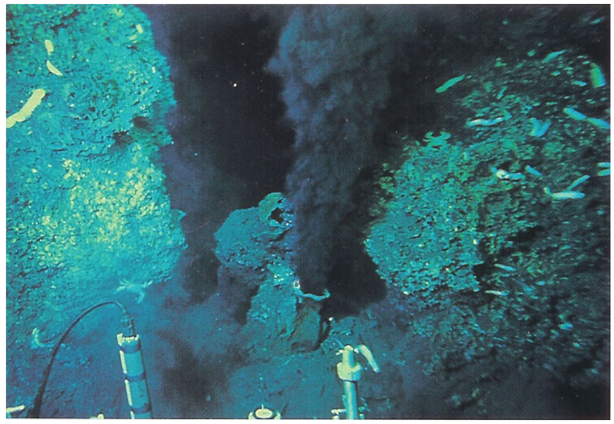

Figure 10.1: A hydrothermal vent at the bottom of the Pacific Ocean, discharging hot water (up to 400°C) and forming a vertical plume. The dark color, giving rise to the nickname *black smoker*, is due the presence of sulfides in the water. [Photograph taken by Dudley Foster, courtesy of the Woods Hole Oceanographic Institution]

$$F \;=\; \frac{\alpha g}{\rho_0 C_p}\, \frac{\text{heat}}{\text{time}}\ . \tag{10.1}$$

Note that because heat per time is expressed in J/s, the buoyancy flux is measured in units of $\mathrm{m}^4/\mathrm{s}^3$.

Let us consider a three-dimensional radially symmetric plume progressing vertically from the bottom through a homogeneous and resting fluid, as shown in Figure 10.2. If we denote by T_0 the temperature of the ambient fluid, then the temperature inside the plume has the value $T_0 + T'$, in which T' denotes the temperature anomaly (positive in a rising plume, negative in a sinking plume). To this temperature anomaly corresponds a density anomaly $\rho' = -\alpha\rho_0 T'$. From the latter, it is convenient to define the local buoyancy, or reduced gravity, g' as:

$$g' \;=\; -\,g\,\frac{\rho'}{\rho_0} \;=\; +\,\alpha g T'. \tag{10.2}$$

Naturally, because of the heterogeneous structure of the plume, with entrainment and dilution taking place along its sides, the buoyancy g' and vertical velocity w within the plume depend on both distance z above the source and radial distance r from the centerline. Like for turbulent jets (previous chapter), observations reveal that the Gaussian profile (bell curve) provides a realistic description of the statistical averages of g' and w over the turbulent fluctuations. Before using such expressions, however, we shall initially limit ourselves to considering only cross-plume averages $\bar{w}$ and $\bar{g}'$, each a function of z, the height within the plume.

The buoyancy flux F can be expressed as the integral across the section of the plume of the product of the vertical velocity w with the buoyancy g'. In terms of

Figure 10.2: A smokestack plume rising in still air. (Photo by the author)

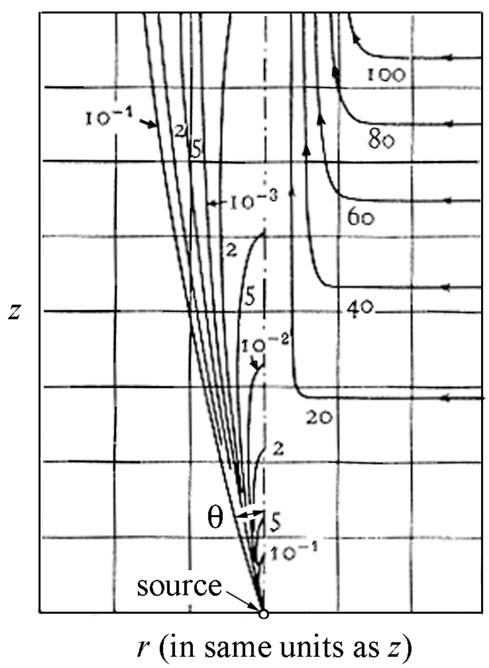

Figure 10.3: Mean isotherms (left) and streamlines (right) in and around a plume maintained from a punctual source at the bottom. The numbers on the isotherms are relative values g'/g and those along the streamlines are relative values of the Stokes streamfunction. (From Rouse et al., 1952 as shown in Turner, 1973)

the mean values across the plume, a reasonably good approximation is

$$F \;=\; \pi R^2 \; \bar{w}\bar{g}' \;=\; \pi R^2 \; \alpha g \bar{T}' \bar{w}, \tag{10.3}$$

where $R(z)$ is the radius of the plume at level z.

As the plume rises, it entrains ambient fluid, but this does not change the heat flux carried by the plume since that ambient fluid carries no heat anomaly. Thus, by virtue of conservation of heat, the buoyancy flux remains unchanged with height and is the same at level z as it was at the start of the plume. In other words, the quantity F is constant. It is what drives the plume, like momentum drives a jet.

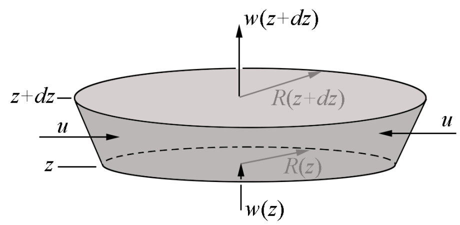

Figure 10.4: A thin slide of the plume on which to perform mass and momentum budgets.

Perfoming a mass conservation budget over a thin slice of the plume extending from level z to level $z + dz$ (Figure 10.4), we can write:

Mass exiting from the top	=	Mass entering through the bottom
	+	Mass entrained through the side.

$$[\pi R^2 \; \rho_0 \bar{w}]_{z+dz} \;=\; [\pi R^2 \; \rho_0 \bar{w}]_z \;+\; 2\pi R dz \; \rho_0 u,$$

in which u is the lateral entrainment velocity and $2\pi R dz$ the lateral area of the slice (Figure 10.4). In differential form, this equation becomes

$$\frac{d}{dz}(R^2 \bar{w}) \;=\; 2Ru. \tag{10.4}$$

Likewise, the vertical momentum budget over the same slice requires

Momentum exiting from the top	=	Momentum entering through the bottom
	+	Momentum entrained through the side
	+	Upward buoyancy force.

There is no vertical momentum acquired by lateral entrainment since the ambient fluid is at rest. By virtue of Archimedes' principle, the upward buoyancy force is equal to the weight of the displaced fluid (at density ρ_0) minus the actual weight of the plume segment (at lower density $\rho_0 + \bar{\rho}'$), for a total of

$$\begin{aligned}\text{Upward buoyancy force} &= \pi R^2\ \rho_0\ g\ dz\ -\ \pi R^2\ (\rho_0 + \bar{\rho'})\ g\ dz \\ &= -\pi R^2\ \bar{\rho'}\ g\ dz \\ &= +\pi R^2\ \rho_0\ \bar{g}'\ dz,\end{aligned}$$

by virtue of (10.2). Thus, the vertical momentum budget takes the form:

$$[\pi R^2\ \rho_0 \bar{w}^2]_{z+dz}\ =\ [\pi R^2\ \rho_0 \bar{w}^2]_z\ +\ \pi R^2\ \rho_0 \bar{g}'\ dz,$$

or, in differential form,

$$\frac{d}{dz}(R^2\ \bar{w}^2)\ =\ R^2\ \bar{g}'. \tag{10.5}$$

Let us stop for a moment and take stock of the equations we have. There are three equations: (10.3) from the heat budget, (10.4) from the mass budget, and (10.5) from the momentum budget. And, there are four unknowns: the radius R, the average velocity $\bar{w}$, the entrainment velocity u, and the averaged buoyancy $\bar{g}'$, each a function of the elevation z. There is thus one more unknown than available equations. To close the problem without solving for the details of the turbulent flow, we state in analogy with the turbulent jet, that the entrainment velocity is proportional to the shear flow induced by the plume. In other words, we assume proportionality between u and $\bar{w}$:

$$u\ =\ a\ \bar{w}, \tag{10.6}$$

with constant dimensionless coefficient a.

The solution of the problem ought to be expressed solely in terms of the quantities F (in $\mathrm{m}^4/\mathrm{s}^3$) and z (in m), because those are the only dimensional variables entering the equations. Thus, dimensional considerations lead us to anticipate the form of the solution:

$$\begin{aligned}R &= \tan\theta\ z \\ \bar{w} &= b\ \frac{F^{1/3}}{z^{1/3}} \\ \bar{g}' &= c\ \frac{F^{2/3}}{z^{5/3}}\ ,\end{aligned}$$

where θ is the angle of the cone made by the plume (Figure 10.3). Observations (Turner, 1973) indicate that this angle is about 8.9°, for which $\tan\theta = 0.157$.

Substitution in the equations at our disposal yields:

$$\begin{aligned}\text{Eq. (10.3)} &\longrightarrow \pi bc \tan\theta\ =\ 1 \\ \text{Eq. (10.4)} &\longrightarrow \frac{5}{3}\ \tan\theta\ =\ 2a \\ \text{Eq. (10.5)} &\longrightarrow \frac{4}{3}\ b^2\ =\ c.\end{aligned}$$

The values of the dimensionless coefficients are found to be

$$
\begin{aligned}
a &= \frac{5}{6} \tan\theta = 0.1305 \\
b &= \left(\frac{3}{4\pi \tan^2\theta} \right)^{1/3} = 2.14 \\
c &= \left(\frac{4}{3\pi^2 \tan^4\theta} \right)^{1/3} = 6.08
\end{aligned}
$$

and the assembled solution is

$$R = 0.157\, z \tag{10.7}$$

$$\bar{w} = 2.14\, \frac{F^{1/3}}{z^{1/3}} \tag{10.8}$$

$$u = 0.130\, \bar{w} = 0.279\, \frac{F^{1/3}}{z^{1/3}} \tag{10.9}$$

$$\bar{g}' = 6.08\, \frac{F^{2/3}}{z^{5/3}} \; . \tag{10.10}$$

Let us now pass from $\bar{w}$ and $\bar{g}'$ averaged across the plume to functions w and g' with Gaussian profile across the plume. For this, we write:

$$w = w_{\max}(z)\ \exp\left(-\frac{r^2}{2\sigma^2} \right) \tag{10.11}$$

$$g' = g'_{\max}(z)\ \exp\left(-\frac{r^2}{2\sigma^2} \right) , \tag{10.12}$$

with the standard deviation $\sigma(z)$ being such that $2\sigma(z)$ represents the radius $R(z)$ of the plume at height z. (See theory for the turbulent jet in the previous chapter.) Thus, $\sigma = R/2$. The peak values along the plume's centerline can then be related to their respective averages by

$$\bar{w} = \frac{1}{\pi R^2} \int_0^\infty w\ 2\pi r dr \tag{10.13}$$

$$\bar{g}' = \frac{1}{\pi R^2} \int_0^\infty g'\ 2\pi r dr, \tag{10.14}$$

and we obtain:

$$w = 4.27\, \frac{F^{1/3}}{z^{1/3}}\ \exp\left(-\frac{81.6 r^2}{z^2} \right) \tag{10.15}$$

$$g' = 12.2\, \frac{F^{2/3}}{z^{5/3}}\ \exp\left(-\frac{81.6 r^2}{z^2} \right) . \tag{10.16}$$

Laboratory experiments (Turner, 1973), indicate that the following adjusted expressions

$$w = 4.7 \frac{F^{1/3}}{z^{1/3}} \exp\left(-\frac{96r^2}{z^2}\right) \quad (10.17)$$

$$g' = 11 \frac{F^{2/3}}{z^{5/3}} \exp\left(-\frac{71r^2}{z^2}\right) \quad (10.18)$$

better match the observations. Note that with these last expressions, the width of the velocity profile is slightly narrower than that of the buoyancy.

10.2 Plume in a Cross-Flow

Line thermal model

10.3 Plume in a Stratified Environment

When a plume rises (or sinks) in a stratified environment, it encounters a temperature becoming closer to its own and progressively loses buoyancy. At some level, it will have lost all buoyancy and will begin to spread horizontally. Such is the case of a smokestack plume in a calm (no wind) and stratified atmosphere typical of the early morning (Figure 10.5). The obvious question to ask is how high does the plume reach?

The stratified ambient fluid is characterized by its stratification frequency N defined from

$$N^2 = \alpha g \frac{dT}{dz} \quad (10.19)$$

where $T(z)$ is the temperature profile of the ambient fluid.

To describe a plume in this type of environment, the same quantities are needed as before, namely the plume's radius R, averaged vertical velocity $\bar{w}$, and averaged buoyancy $\bar{g}'$, each function of the elevation z. The difference with the previous section is that now the buoyancy flux F, defined in (10.3), is no longer a constant along the axis of the plume. Three equations are at our disposal: The mass budget

$$\frac{d}{dz}(\bar{w}R^2) = 2uR = 2a\bar{w}R, \quad (10.20)$$

the momentum budget

$$\frac{d}{dz}(\bar{w}^2 R^2) = \bar{g}' R^2, \quad (10.21)$$

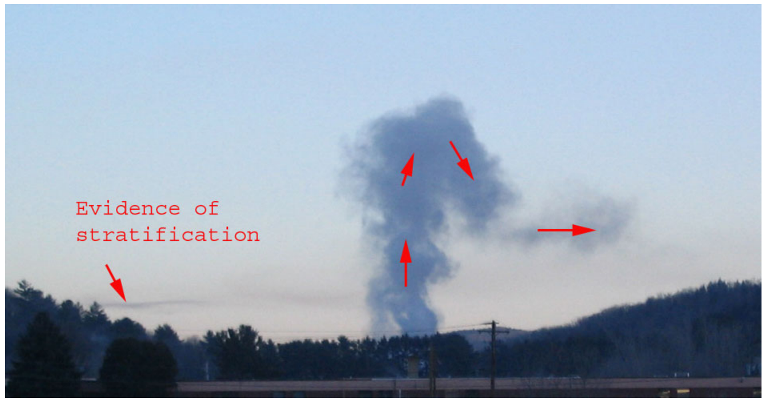

Figure 10.5: A plume rising in an early morning when the lower atmosphere is stratified. A clue of this stratification is the thin horizontal band of cloud on the left of the plume (marked in picture). Note the inertial overshoot of the plume cloud before settling at the level of neutral buoyancy. [Photograph by the author]

which remain unchanged, and the heat budget, which is

$$\frac{d}{dz}(\bar{w}\bar{T}_{\text{plume}}\ \pi R^2) \ = \ uT(z)\ 2\pi R. \tag{10.22}$$

Using the buoyancy locally experienced by the plume, $\bar{g}' = \alpha g[\bar{T}_{\text{plume}} - T(z)]$, the last equation can be recast as

$$\frac{d}{dz}(\bar{g}'\bar{w}R^2) \ + \ \frac{d}{dz}[\alpha g T(z)\ \bar{w}R^2] \ = \ 2uR\ \alpha g T(z).$$

With Equation (10.20) and definition $\alpha g\ dT(z)/dz = N^2$, it can be reduced to

$$\frac{d}{dz}(\bar{g}'\bar{w}R^2) \ = \ -N^2\ \bar{w}\ R^2. \tag{10.23}$$

The solution to this set of equations does not exhibit similarity, but the equations can easily be integrated numerically. Starting with initial conditions (at level $z = 0$) such that the momentum and volumetric flow are nil but buoyancy flux finite, one finds the results shown in Figure 10.6. On that figure, the variables are made dimensionless by scaling as follows: z and R by $(F/N^3)^{1/4}$, $\bar{w}$ by $(F/N)^{1/4}$, and $\bar{g}'$ by $(F/N^5)^{1/4}$, where F is the starting buoyancy flux (at $z = 0$). The entrainment parameter a was taken as 0.125. (For a similar numerical integration, see Morton, Taylor and Turner, 1956).

We note on Figure 10.6 that the buoyancy crosses zero at $z = 2.98(F/N^3)^{1/4}$, but the residual vertical velocity at that level makes the plume overshoot, up to $z = 3.92\ (F/N^3)^{1/4}$, by which level the vertical velocity vanishes and the radius becomes infinite.

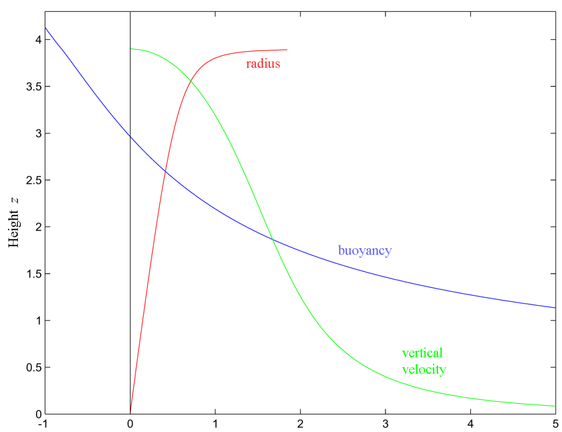

Figure 10.6: Numerical integration of Equations (10.20), (10.21) and (10.23), with $a = 0.125$, tracing the vertical structure of a buoyant plume as it rises in a stratified environment. The buoyancy crosses zero at $z = 2.98$, above which the plume becomes negatively buoyant. The vertical velocity vanishes at $z = 3.92$ and the radius becomes infinite. See text for details of the non-dimensionalization employed in making the graph.

Laboratory experiments and field observations confirm and tweak this theoretical prediction (Figure 10.7). Briggs (1969) gives

$$z_{\max} \;=\; 5.0\, \left(\frac{F}{\pi N^3}\right)^{1/4} \;=\; 3.76\, \left(\frac{F}{N^3}\right)^{1/4}. \qquad (10.24)$$

10.4 Thermals

A thermal is a finite parcel of fluid consisting of the same fluid as its surroundings but at a different temperature. Because of its buoyancy, a cold thermal sinks (negative buoyancy), while a warm thermal rises (positive buoyancy). The name was given by glider pilots to what they perceived as regions of warm air rising above a heated ground in which they could soar. Convection in the atmosphere does indeed proceed by means of rising thermals (Priestley, 1959). The situation, however, can be quite chaotic, with a collection of thermals rising here and there at various times, some of them smaller and slower, and others larger and faster. Here, for the sake of understanding the basic mechanism, we shall be concerned with a single thermal immersed in an infinite homogeneous fluid at rest.

Experiments have been conducted in the laboratory (Figure 10.8), and it has been found that all thermals roughly behave in similar ways: as they rise (or sink), they entrain surrounding fluid and become more dilute, thereby slowing down in their ascent (or descent). The actual shape of a thermal, however, can vary considerably from one set of observations to another. Here, basic dynamics supplemented by a few dimensionless numbers gleaned from experiments will be used to establish a simple theory for the prediction of a thermal's behavior over time.

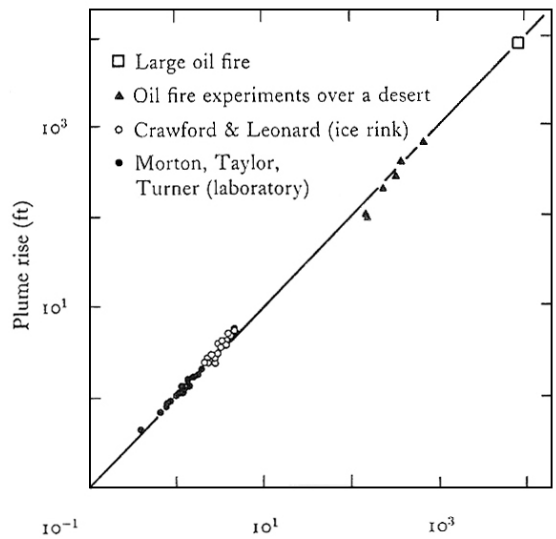

Figure 10.7: Measurements of plume rise in calm stratified surroundings, revealing that the ultimate height reached by a plume follows Equation (10.24). (Adapted from Briggs, 1969)

The key property of a thermal is its total buoyancy, defined as

$$B \ = \ \alpha g T' V \ = \ g' V \tag{10.25}$$

in which V is the volume of the thermal, T' its temperature anomaly, and $g' = \alpha g T'$ the reduced gravity it experiences. This total buoyancy is a conserved quantity as the thermal rises (or sinks) because, while it entrains surrounding fluid, its temperature anomaly decreases by dilution in proportion to its volume increase, thus keeping the product $T'V$ constant during the thermal's life.

The volume of a thermal can be expressed as

$$V \ = \ mR^3, \tag{10.26}$$

where R is the radius of the thermal seen from above, and m is a coefficient less that $4\pi/3 = 4.2$ (value for a spherical volume) because a thermal has a slightly flattened shape. The value of m is notoriously difficult to measure, and some indirect measurement is in order, as we shall see later.

Mass conservation over time can be expressed as

$$\frac{dV}{dt} \ = \ Au,$$

in which A is the enclosing surface area of the thermal and u the average entrainment velocity across that surface (Figure 10.9). Taking the area A as proportional to R^2, the square of the thermal's radius, and the entrainment velocity u as proportional to the thermal's vertical velocity w, we can express the preceding equation as

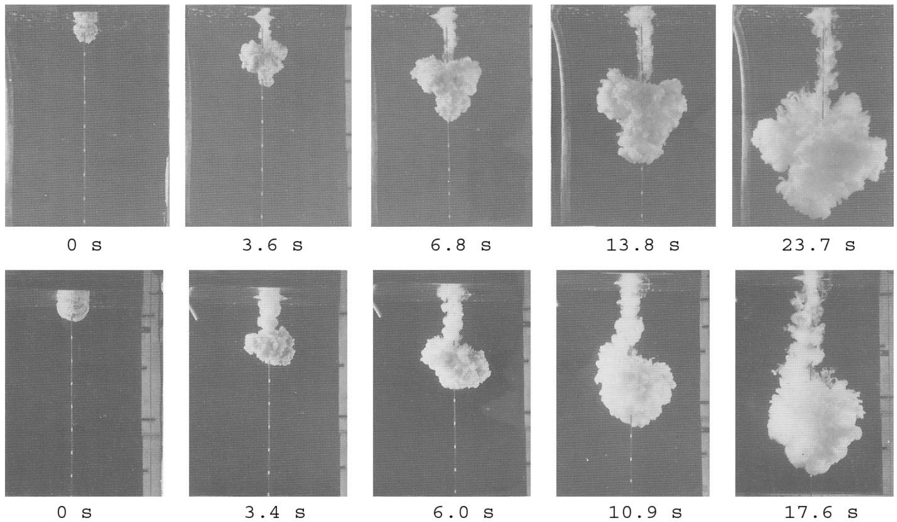

Figure 10.8: Descending thermals in a laboratory experiment. These thermals in water are made visible by barium sulfate. The stem left behind by each thermal is due to the manner a spherical cap was rotated to provoke the release. The second thermal (bottom row) has a larger negative buoyancy than the first (top row). [From Scorer, 1997]

$$\frac{dV}{dt} = aR^2 w, \tag{10.27}$$

in which the coefficient a ought to be a dimensionless constant, to be determined from experiments or observations. Using (10.26), this equation can be reduced to:

$$\frac{dR}{dt} = \frac{a}{3m}\, w. \tag{10.28}$$

The momentum budget over time takes the form

$$\begin{aligned}
\frac{d}{dt}\left(\frac{3}{2}\rho_{\text{thermal}} V w\right) &= \text{Upward buoyancy force } - \text{ Downward weight} \\
&= \rho_{\text{ambient}}\, Vg \;-\; \rho_{\text{thermal}}\, Vg \\
&= \rho_{\text{thermal}}\, \alpha T'\, Vg \\
&= \rho_{\text{thermal}}\, g' V,
\end{aligned}$$

in which the factor 3/2 on the left is due to the *added-mass effect*. Physically, the thermal is subject to its own acceleration (time derivative of one time $\rho_{\text{thermal}} V w$), but its changing pace also causes acceleration of the surrounding fluid diverted by

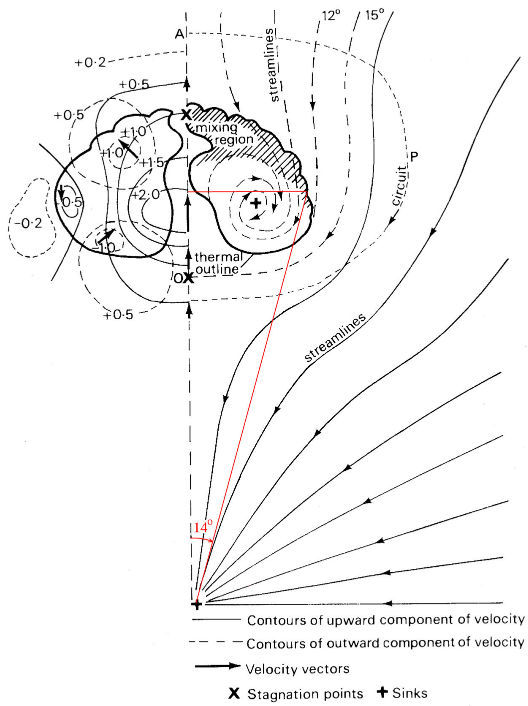

Figure 10.9: The anatomy of a rising thermal according to Scorer (1997). Fluid within a cone of about 12° is entrained by the top of the thermal, while fluid oustide of this and within a wider cone of 15° is entrained in the rear. The rest of the ambient fluid is merely deflected by the passage of the thermal. The oblique red line traces the outer edge of the thermal over time, forming an angle of about 14° from the vertical. (From Scorer, 1997)

its passage, effectively accelerating 50% more fluid mass, hence the factor $3/2 = 1.5$. Division by ρ_{thermal}, which at all times remains close to the reference density of the fluid, yields

$$\frac{d}{dt}(Vw) \;=\; \frac{2}{3}\, g'V. \tag{10.29}$$

Elimination of the product $g'V$ by virtue of Equation (10.25) indicates that the right-hand side of the preceding equation is a constant, leading to an immediate integration:

$$Vw \;=\; \frac{2}{3}\, B\, t, \tag{10.30}$$

for which $t = 0$ marks the time when the thermal had zero momentum. Next, solving for w ($w = 2Bt/3V = 2Bt/3mR^3$) and replacing in Equation (10.28), we obtain a single equation for the radius R of the thermal:

$$\frac{dR}{dt} \;=\; \frac{2a}{9m^2}\,\frac{B\,t}{R^3}\;.$$

The solution of this equation is:

$$R = \left(\frac{4a}{9m^2}\right)^{1/4} B^{1/4}\ t^{1/2}. \tag{10.31}$$

Now knowing the radius as a function of time, we can readily solve for the other quantities, namely volume V, vertical velocity w and reduced gravity g':

$$V = \left(\frac{64a^3}{729m^2}\right)^{1/4} B^{3/4}\ t^{3/2} \tag{10.32}$$

$$w = \left(\frac{9m^2}{4a^3}\right)^{1/4} \frac{B^{1/4}}{t^{1/2}} \tag{10.33}$$

$$g' = \left(\frac{729m^2}{64a^3}\right)^{1/4} \frac{B^{1/4}}{t^{3/2}}\ . \tag{10.34}$$

In these expressions, it is clear that the time origin actually refers to a virtual stage in which the thermal had zero volume, infinite velocity and infinite temperature anomaly. Obviously, the actual life of the thermal started some finite time after this, with a finite volume, finite velocity and finite temperature anomaly.

Note that the complete solution depends on two dimensionless parameters, a and m. Since neither is easy to determine directly, it is wise to seek their value indirectly by matching thermal's properties that are more readily observed. One such property is the manner in which the thermal's radius grows with distance. For this, we integrate $dz/dt = w$ to obtain the thermal's elevation as a function of time. The result is:

$$z = \left(\frac{36m^2}{a^3}\right)^{1/4} B^{1/4}\ t^{1/2}. \tag{10.35}$$

It appears that both elevation z and radius R grow at similar rates, yielding a constant ratio:

$$\frac{R}{z} = \frac{a}{3m}\ . \tag{10.36}$$

Laboratory observations (Figure 10.9) reveal that this is indeed the case that thermals behave in a self-similar way, and that the ratio of R to z is about $\tan 14^\circ = 0.25$. Thus,

$$R = 0.25\ z, \tag{10.37}$$

and $a = 0.75m$.

The other reliable observation is that the ratio z^2/t (a time constant as predicted by the theory) varies from experiment to experiment in proportion to $\sqrt{B}$ (Figure 10.10). The theoretical coefficient of proportionality is $\sqrt{36m^2/a^3}$, and experiments give it a value of 5.80. Solving $a = 0.75m$ together with $\sqrt{36m^2/a^3} = 5.80$ yields: $a = 1.90$ and $m = 2.54$. From this follow all other coefficients:

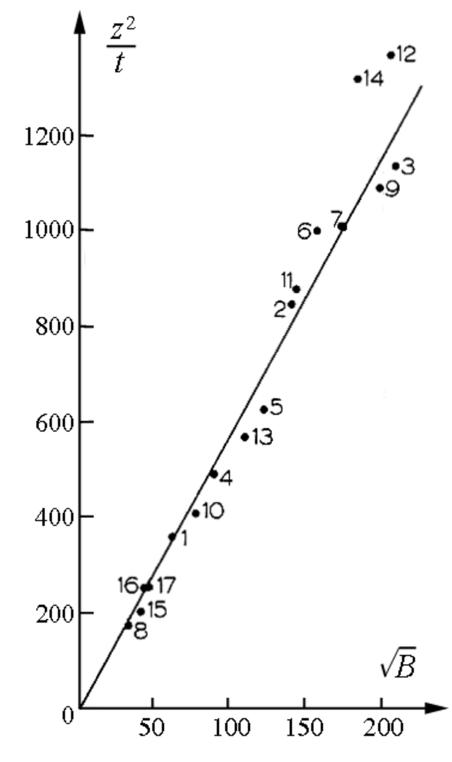

Figure 10.10: Plot of the quantity z^2/t (a time constant during the life of thermal) versus the square root of the thermal's buoyancy. Each numbered dot refers to a different laboratory experiment, and the solid line shows the best linear fit. (From Scorer, 1997)

$$R = 0.60\ B^{1/4}\ t^{1/2} \tag{10.38}$$

$$V = 0.55\ B^{3/4}\ t^{3/2} \tag{10.39}$$

$$w = 1.20\ \frac{B^{1/4}}{t^{1/2}} \tag{10.40}$$

$$z = 2.41\ B^{1/4}\ t^{1/2} \tag{10.41}$$

$$g' = 1.81\ \frac{B^{1/4}}{t^{3/2}}. \tag{10.42}$$

10.5 Thermals in a Stratified Environment

When a thermal rises (or sinks) in a stratified environment, it progressively encounters a temperature closer to its own and therefore loses its buoyancy. Ultimately, it will reach a level of no buoyancy and begin to spread laterally (Figure 10.11). With no thermal contrast left, the thermal loses its identity. What is this ultimate level is not an obvious question. Indeed, it can be easily established that the thermal will never reach the level of its initial temperature. The reason is its partial dilution by entrainment of surrounding fluid (which changes as the thermal crosses isotherms) and its consequent dilution.

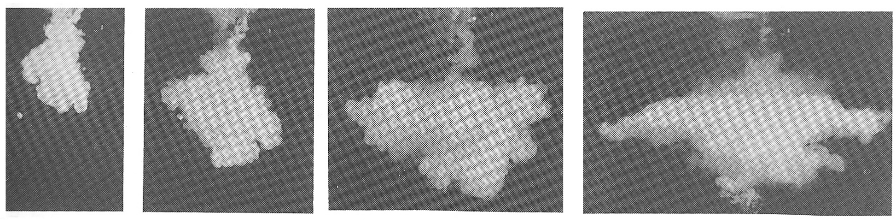

Figure 10.11: A laboratory experiment of a thermal sinking in a stratified environment. Note the ultimate arrest and spreading of the thermal once it loses its buoyancy. (From Scorer, 1997)

A practical application of this situation is the dumping of waste in a stratified body of water: The dumped waste sinks from the surface, gradually mixes with surrounding water during its fall, and eventually settles down at some intermediate depth. The determination of that depth is crucial in water quality studies and permitting

The stratified environment is characterized by its stratification frequency N defined from

$$N^2 \;=\; \alpha g \, \frac{dT}{dz} \; , \tag{10.43}$$

where $T(z)$ is the temperature profile of the ambient fluid.

To track a thermal in this environment, the same quantities are needed as before, namely the thermal's radius R, volume $V = mR^3$, vertical position z, vertical velocity $w = dz/dt$, reduced gravity g', and total buoyancy $B = g'V$. The difference with the previous section is that now the total buoyancy B is no longer a constant of the motion. Three equations are at our disposal: The mass budget

$$\frac{dV}{dt} \;=\; Au \;=\; aR^2 w, \tag{10.44}$$

the momentum budget

$$\frac{d}{dt}(Vw) \;=\; \frac{2}{3}\, g'V, \tag{10.45}$$

which remain unchanged, and the heat budget, which is

$$\frac{d}{dt}(VT_{\mathrm{thermal}}) \;=\; AuT(z) \;=\; aR^2 wT(z). \tag{10.46}$$

Using the reduced gravity locally experienced by the thermal, $g' = \alpha g [T_{\text{thermal}} - T(z)]$, the last equation can be recast as

$$\frac{d}{dt}(Vg') \ + \ \frac{d}{dt}[V \ \alpha g T(z)] \ = \ aR^2 w \ \alpha g T(z).$$

Using (10.44) and the fact that $dT(z)/dt = (dT/dz) \times (dz/dt) = w \ (dT/dz)$, it reduces to

$$\frac{d}{dt}(Vg') \ = \ - N^2 V w. \tag{10.47}$$

Eliminating V from Equations (10.44), (10.45) and (10.47) by using $V = mR^3$ yields a set of three equations for the three unknowns R, w and g':

$$\frac{dR}{dt} \ = \ \frac{a}{3m} \ w \tag{10.48}$$

$$\frac{d}{dt}(R^3 w) \ = \ \frac{2}{3} \ R^3 g' \tag{10.49}$$

$$\frac{d}{dt}(R^3 g') \ = \ -N^2 R^3 w. \tag{10.50}$$

Viewing these three quantities no longer as functions of time t but rather of elevation z and calling upon $w = dz/dt$, we can transform them into:

$$\frac{dR}{dz} \ = \ \frac{a}{3m} \tag{10.51}$$

$$\frac{d}{dz}(R^3 w) \ = \ \frac{2R^3 g'}{3w} \tag{10.52}$$

$$\frac{d}{dz}(R^3 g') \ = \ -N^2 R^3. \tag{10.53}$$

The first of these equations yields

$$R \ = \ R_0 + \frac{a}{3m} z, \tag{10.54}$$

in which R_0 is the initial radius at the departure level $z = 0$. As long as N^2 is a constant (linear stratification), the last equation of the set can, too, be integrated to yield:

$$R^3 g' \ = \ R_0^3 g_0' \ - \ \frac{3m}{4a} \ N^2 \ (R^4 - R_0^4) \tag{10.55}$$

in which g_0' are is the initial value of g'.

The thermal loses all its buoyancy when its g' drops to zero, which occurs when its radius has grown to the value R_{end} such that

$$\frac{3m}{4a} \ N^2 \ (R_{\text{end}}^4 - R_0^4) \ = \ R_0^3 g_0', \tag{10.56}$$

that is

$$R_{\text{end}} = \left(R_0^4 + \frac{4a}{3m} \frac{R_0^3 g_0'}{N^2} \right)^{1/4} . \tag{10.57}$$

Assuming that the thermal started with an insignificant radius and has vastly expanded during its travel, we can approximate the preceding expression to

$$\begin{aligned} R_{\text{end}} &\approx \left(\frac{4a}{3m} \frac{R_0^3 g_0'}{N^2} \right)^{1/4} \\ &\approx \left(\frac{4a}{3m^2} \frac{B_0}{N^2} \right)^{1/4} , \end{aligned} \tag{10.58}$$

where $B_0 = V_0 g_0' = m R_0^3 g_0'$ is the thermal's initial buoyancy. Translating this radius into the corresponding elevation gives the terminal level where the thermal loses its identity:

$$\begin{aligned} z_{\text{end}} &= \frac{3m}{a} (R_{\text{end}} - R_0) \approx \frac{3m}{a} R_{\text{end}} \\ &\approx \left(\frac{108 m^2}{a^3} \right)^{1/4} \left(\frac{B_0}{N^2} \right)^{1/4} . \end{aligned} \tag{10.59}$$

With the parameter values $a = 1.90$ and $m = 2.54$ determined at the end of the previous section, we have

$$z_{\text{end}} \approx 3.17 \left(\frac{B_0}{N^2} \right)^{1/4} . \tag{10.60}$$

Note that at the level where $g' = 0$, the thermal has some residual vertical velocity and will overshoot slightly its level of neutral buoyancy. This explains the bulge on the front side of the thermal seen in Figure 10.11.

10.6 Buoyant Puffs

Text

10.7 Gravity Flows

We close this chapter by examining buoyancy-driven flows along boundaries. The simplest case is that solved by Benjamin (1968) and illustrated in Figure 10.12: A

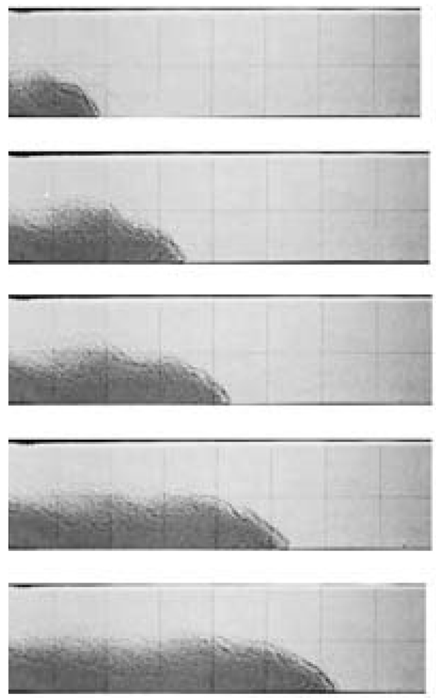

Figure 10.12: A dense bottom current intruding under an initially resting fluid of lower density. The numbers on the right indicate a dimensionless time. (*Source*: Shin *et al.*, 2004, segment of Fig. 2)

lighter fluid initially at rest between two frictionless horizontal boundaries is being partially displaced by an intruding denser fluid that progresses along the bottom boundary.

Such a situation may be modeled as depicted in the top panel of Figure 10.13 with a top fluid of density ρ_0 at rest ahead of an intrusion of slightly higher density $\rho_0 + \Delta\rho$ ($\Delta\rho << \rho_0$). The height of the domain is denoted H, and the intrusion occupies in its wake a height h ($< H$). Laboratory experiments (Shin *et al.*, 2004) indicate that the intrusion progresses at a constant speed, which we denote u_2, and we subtract this velocity from all velocities in order to work in a reference frame with the intruding flow at rest, as depicted in the lower panel of Figure 10.13. We invoke the Boussinesq approximation based on $\Delta\rho << \rho_0$, assume frictionless boundaries, and ignore any mixing between the two fluids despite the velocity shear between them. Our goal is the determination of the intrusion speed u_2 in terms of the thickness h of its wake.

The analysis begins with mass conservation:

$$\rho_0(H-h)\,(u_1-u_2) \;=\; -\,\rho_0 H u_2\;,$$

which provides u_1 in terms of u_2:

$$u_1 \;=\; -\,\frac{h}{H-h}\,u_2\;. \tag{10.61}$$

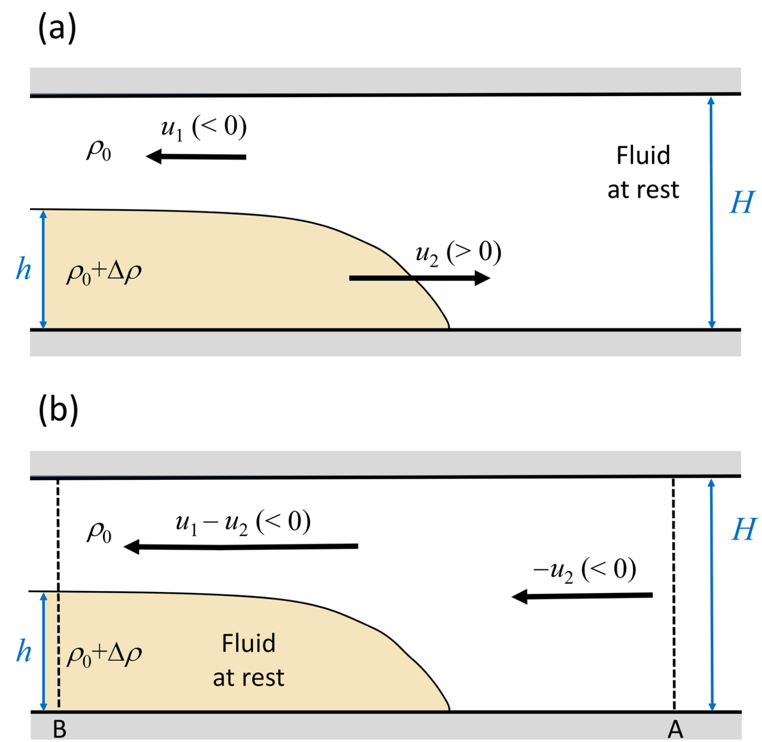

Figure 10.13: Schematic diagram of a dense intrusion in an otherwise quiescent fluid. *Top panel*: In the reference frame with fixed boundaries. *Bottom panel*: In the reference frame moving with the intrusion, in which the flow is steady.

Next, in preparation of a momentum budget, we apply the Bernoulli Principle along the top boundary, which is valid under the assumption of a frictionless boundary. Following the lighter fluid from Section **A** ahead of the intrusion to Section **B** in the wake of the intrusion. we get:

$$p_{top,A} + \frac{1}{2}\rho_0 u_2^2 \;=\; p_{top,B} + \frac{1}{2}\rho_0 (u_1 - u_2)^2 \ ,$$

from which we deduce the drop in pressure along the top boundary caused by the passage of the intrusion:

$$p_{top,B} - p_{top,A} \;=\; -\frac{1}{2}\rho_0 u_1 \,(u_1 - 2u_2) \;<\; 0 \ . \qquad (10.62)$$

By considering the pressures at both sections, from the top down and integrating vertically, we obtain the pressure forces acting on the system:

$$
\begin{aligned}
0 \le z \le H: \quad p_A(z) &= p_{top,A} + \rho_0 g \ (H - z) \\
h \le z \le H: \quad p_B(z) &= p_{top,B} + \rho_0 g \ (H - z) \\
0 \le z \le h: \quad p_B(z) &= p_{top,B} + \rho_0 g \ (H - h) + \Delta\rho g \ (h - z) \ ,
\end{aligned}
$$

to yield the following pressure forces:

$$
\begin{aligned}
\int_0^H p_A(z) \ dz &= p_{top,A} H + \frac{1}{2}\rho_0 g H^2 \\
\int_0^H p_B(z) \ dz &= p_{top,B} H + \frac{1}{2}\rho_0 g H^2 + \frac{1}{2}\Delta\rho g h^2 \ .
\end{aligned}
$$

The subtraction of these two with use of (10.62) gives the net pressure force exerted on the system from left to right between Sections **A** and **B**:

$$
F_p \ = \ \int_0^H p_B(z) \ dz - \int_0^H p_A(z) \ dz \ = \ \frac{1}{2} g \Delta\rho h^2 \ - \ \frac{1}{2}\rho_0 H u_1 (u_1 - 2u_2) \ . \quad (10.63)
$$

We are now in the position to express the momentum balance between Sections **A** and **B**. This balance states that the exiting momentum at **B** is equal to the entering momentum at **A** minus the opposing pressure force pushing from left to right:

$$
\rho_0 (H - h) \ (u_1 - u_2)^2 \ = \ \rho_0 H u_2^2 \ - \ F_p
$$

which yields

$$
\frac{1}{2}\rho_0 H u_1 (2u_2 - u_1) \ + \ \rho_0 h (u_1 - u_2)^2 \ = \ \frac{1}{2}\Delta\rho g h^2 \ . \quad (10.64)
$$

The final step is to eliminate the velocity u_1 to obtain an equation solely for the intrusion speed u_2. For this, we invoke the earlier volumetric flow budget (10.61), and we obtain:

$$
u_2 \ = \ \sqrt{g \frac{\Delta\rho}{\rho_0} \frac{(H - h)^2}{H}} \ . \quad (10.65)
$$

Cite paper by Eckart Meiburg on his use of the (Bjerknes) circulation theorem.

Problems

10-1. By using a blower and some preheating, one can adjust both the upward velocity and buoyancy of fumes exiting from the top of a smokestack. Specifically, two scenarios are being considered, one with more velocity and one with more buoyancy, as follows:
Scenario 1: Average exit vertical velocity = 12 m/s
Average exit buoyancy = 0.01 m/s^2
Scenario 2: Average exit vertical velocity = 1 m/s
Average exit buoyancy = 0.12 m/s^2.
In each case, the exit diameter is 1.5 m and the entrainment coefficient a is taken as 0.115.
Which of the two scenarios gives the highest vertical velocity at the center of the plume 20 m above the smokestack?

10-2. You notice a buzzard soaring in a circling fashion and guess that it is taking advantage of the upward motion of a thermal. As you happen to have meteorological gear with you, including a radar profiler, you determine that the buzzard is flying at an altitude of 80 m and that the temperature at the center of the bird's circle is 0.30°C higher than outside the thermal, where the temperature is 25°C.
What is the radius of the thermal and its center vertical velocity? Also, how old is this thermal?

10-3.

10-4.

10-5. Show that, for a thermal rising in a homogeneous ambient fluid, w^2 is equal to $g'z/2$. Does this relation have any particular significance?

10-6. Establish the form of the total energy of a thermal (kinetic plus potential) for a thermal rising in a uniform environment and determine its variation along the path of the thermal. Resolve any paradox.

10-7. It was mentioned at the end of the section on thermals in a stratified environment that, once it reaches its level of neutral buoyancy, a thermal still possesses a residual vertical velocity. What is that velocity? And, at what ultimate level z does the vertical velocity finally vanish? Assume that the initial radius of the thermal was negligible compared to its radius at the level of neutral buoyancy.

10-8.

10-9.