Chapter 3

Differential Equations for Fluid Motion

SUMMARY: In this chapter, we derive the partial-differential equations that govern fluid motion and make a series of simplifications and modifications to adapt them to environmental applications. The differential approach is exposed in addition to the control-volume approach of the previous chapter because it is better suited to certain applications, particularly the study of waves and instabilities. The differential approach is also used in the development of most numerical models.

3.1 Equations of Motion

3.1.1 Budget for an infinitesimal volume

The finite-volume approach of the previous chapter fails when it is important to recognize that properties of the fluid vary locally. For example, the wind above the ground surface varies gradually with height and, in water wave motion, the velocity not only changes in space but also with time. In such instances and many others, a continuous representation of the fluid is necessary and, to obtain the requisite equations, we apply the analysis of the previous chapter to an infinitesimal volume element $V = \Delta x \Delta y \Delta z$ (Figure 3.1) and subsequently shrink this volume to a point by taking the limit $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$.

This parallelepiped volume has six sides and is therefore subject to six distinct fluxes of the arbitrary quantity c. On the left and right sides, the quantity c enters and leaves with the x-component u of the velocity, and it is necessary to distinguish between values of c and u on the left (at $x - \Delta x/2$) from those on the right (at $x + \Delta x/2$). Similarly, there is pair of fluxes entering and leaving through the front and back sides due to the y-component v of the velocity (at $y \pm \Delta y/2$) and a pair of fluxes entering and leaving through the bottom and top sides due to the

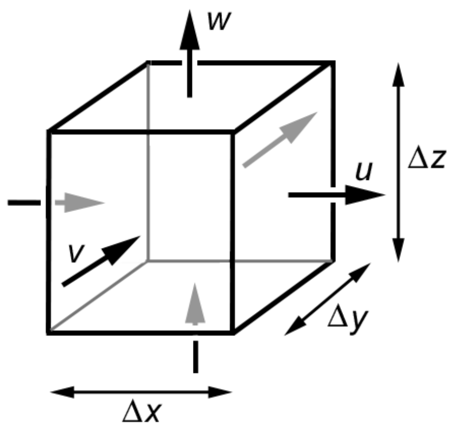

Figure 3.1: A small control volume for a local budget. This volume has six boundaries, two in every of the three directions of space, and the budget includes a total of six import/export fluxes, in addition to a possible internal source.

z-component w of the velocity (at $z \pm \Delta z/2$). These fluxes together with a possible source S inside the small volume leads to the following budget [see Equation (2.8)]:

$$
\begin{aligned}
\Delta x \Delta y \Delta z \, \frac{dc}{dt} = \; & + \; c\left(x - \frac{\Delta x}{2}, y, z\right) \; u\left(x - \frac{\Delta x}{2}, y, z\right) \; \Delta y \Delta z \\
& - \; c\left(x + \frac{\Delta x}{2}, y, z\right) \; u\left(x + \frac{\Delta x}{2}, y, z\right) \; \Delta y \Delta z \\
& + \; c\left(x, y - \frac{\Delta y}{2}, z\right) \; v\left(x, y - \frac{\Delta y}{2}, z\right) \; \Delta x \Delta z \\
& - \; c\left(x, y + \frac{\Delta y}{2}, z\right) \; v\left(x, y + \frac{\Delta y}{2}, z\right) \; \Delta x \Delta z \\
& + \; c\left(x, y, z - \frac{\Delta z}{2}\right) \; w\left(x, y, z - \frac{\Delta z}{2}\right) \; \Delta x \Delta y \\
& - \; c\left(x, y, z + \frac{\Delta z}{2}\right) \; w\left(x, y, z + \frac{\Delta z}{2}\right) \; \Delta x \Delta y \\
& + \; S.
\end{aligned}
$$

Dividing throughout by $\Delta x \Delta y \Delta z$ and placing the flux terms on the left-hand side of the equation, we obtain:

$$
\begin{aligned}
\frac{dc}{dt} \; & + \; \frac{(cu)_{x+\Delta x/2} - (cu)_{x-\Delta x/2}}{\Delta x} \\
& + \; \frac{(cv)_{y+\Delta y/2} - (cv)_{y-\Delta y/2}}{\Delta y} \\
& + \; \frac{(cw)_{z+\Delta z/2} - (cw)_{z-\Delta z/2}}{\Delta z} \; = \; \frac{S}{\Delta x \Delta y \Delta z} \; ,
\end{aligned}
$$

and, in the limit $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$, the spatial differences become spatial derivatives:

$$\frac{\partial c}{\partial t} + \frac{\partial (cu)}{\partial x} + \frac{\partial (cv)}{\partial y} + \frac{\partial (cw)}{\partial z} = s, \tag{3.1}$$

where s is the source of the quantity c per unit volume and unit time.

Example 3.1

Consider a pollutant that degrades by participating in chemical reactions. In such case, the 'source' term s is properly speaking a 'sink' and has a negative value. Under first-order chemical kinetics, the rate of removal of the chemical is proportional to its own concentration, because the more chemical is present, the more reactions take place. Thus, a realistic model is $s = -Kc$, where K is a constant of decay, with dimension equal to the inverse of time (*example*: $K = 0.2$/day).

Consider now a one-dimensional system with uniform and constant velocity u. Equation (3.1) becomes:

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = - Kc. \tag{3.2}$$

To solve this equation, we need to know the initial concentration of the pollutant. Let us denote it as $c(x, t = 0) = c_0(x)$.

If there were no chemical degradation ($K = 0$), this patch would simply move downstream without change, and the solution would be $c(x, t) = c_0(x - ut)$, which corresponds to a mere translation down the x–axis by the traveled distance ut. On the other hand, if there were no movement ($u = 0$) and only chemical degradation, the solution would be $c(x, t) = \exp(-Kt)\ c_0(x)$, which is the initial distribution attenuated over time. Combining these two limiting solutions, it is not difficult to show that the distribution of concentration corresponding to the combined case of advection and degradation at any later time t is given by

$$c(x, t) = \mathrm{e}^{-Kt}\ c_0(x - ut). \tag{3.3}$$

As an application, let us consider an accidental release of benzene (C_6H_6, a carcinogenic substance) in a river moving at the uniform speed of 15 cm/s. From water, benzene volatilizes into the air, at a rate that corresponds to a decay constant of 0.20/day (a function of water depth). If the initial concentration reaches a maximum of 25 ppb (= 0.025 mg/L) at the center of the spill, and the drinking standard is 5 ppb (= 0.005 mg/L), which downstream length of the river will be subjected at some time or other to a concentration exceeding the drinking standard and how long will the episode last?

To answer these questions, we track the maximum of the concentration over time. Equation (3.3) tells that the maximum at time t is the initial maximum times the factor $\exp(-Kt)$. The contamination episode ends when the maximum concentration falls to the drinking standard and thus at time t such that (5 ppb) $= \exp(-Kt) \times$ (25 ppb). The solution is $\exp(-Kt) = 5/25 = 0.20$, giving $Kt =$

1.609 and $t = 8.047$ days $= 6.953 \times 10^5$ s. Over this time, the river water has traveled a distance $ut = (0.15$ m/s$)(6.953 \times 10^5$ s$) = 104.3$ km. Thus, more than 100 kilometers of river are being contaminated, and the episode will last slightly more than 8 days.

3.1.2 Conservation of mass

For the mass budget, the concentration c is mass per volume, i.e. density, noted ρ. Since mass cannot be gained or lost in a system, the source term s is zero, and budget (3.1) becomes:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0. \tag{3.4}$$

As noted in Section 1.3 and again in 2.5, environmental applications deal exclusively with air and water at pressures and temperatures staying near ambient values and generating only modest changes in density. Therefore, the density ρ of the fluid may be assimilated to a constant reference value ρ_0 (as if the fluid were incompressible), and the preceding equation may be reduced to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{3.5}$$

This equation is often called the *continuity equation* because it states that the fluid occupies space in a continuous manner, neither leaving holes or occupying the same volume more than once.

In cylindrical coordinates (with velocity components u in the radial r-direction v in the azimuthal θ-direction, and w in the z-direction), the preceding equation becomes

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0. \tag{3.6}$$

Example 3.2

In water above a flat horizontal bottom, a wave propagates in the x-direction with the associated horizontal flow field:

$$u = U \sin(kx - \omega t) , \quad v = 0,$$

where U is the velocity amplitude, $2\pi/k$ the wavelength and $2\pi/\omega$ the period. The question is: What is the vertical velocity?

Since water is incompressible, conservation of mass reduces to the equation of continuity (3.5), which for $v = 0$ becomes

$$\begin{aligned} \frac{\partial w}{\partial z} &= -\frac{\partial u}{\partial x} \\ &= -kU\cos(kx - \omega t). \end{aligned}$$

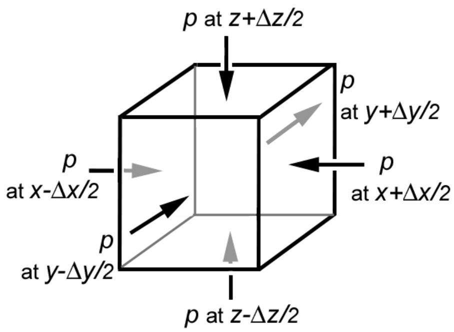

Figure 3.2: Pressure forces on a small control volume as a piece of the local momentum budget.

For a flat and impermeable bottom, at $z = 0$, this equation can be integrated, providing:

$$w \;=\; -\,kUz\;\cos(kx - \omega t).$$

Thus, at some level H above the bottom, the vertical velocity exhibits oscillations of amplitude kUH, which is not equal to that of the horizontal velocity. (Particle trajectories are ellipses, not circles.)

3.1.3 Momentum budget

Next comes momentum, and the quantity c becomes momentum per unit volume, $\rho\vec{u}$, which is a vector. According to Newton's second law, forces act as sources of momentum. As in Section 2.3, we need to consider the two primary forces acting on an environmental fluid parcel, which are pressure and gravity. Under the rubrique of 'other forces', we include here the frictional forces that the fluid parcel experiences in contact with its neighbors.

In the x-direction, there are two pressure forces of the type $p\Delta y\Delta z$, one on the left (at $x - \Delta x/2$) acting in the positive x-direction and the other on the right (at $x + \Delta x/2$) acting in the negative x-direction (Figure 3.2). The sum of the two is:

$$\left[p\left(x - \frac{\Delta x}{2}, y, z\right) \;-\; p\left(x + \frac{\Delta x}{2}, y, z\right)\right]\Delta y\Delta z.$$

A Taylor expansion of the function p near the central point x yields:

$$\text{pressure force in } x\text{-direction} \;=\; -\,\frac{\partial p}{\partial x}\,\Delta x\Delta y\Delta z \;+\; O(\Delta x^2\Delta y\Delta z).$$

Similarly, the components of the pressure force in the y- and z-directions are:

$$\text{pressure force in } y\text{-direction} \;=\; -\,\frac{\partial p}{\partial y}\,\Delta x\Delta y\Delta z \;+\; O(\Delta x\Delta y^2\Delta z)$$

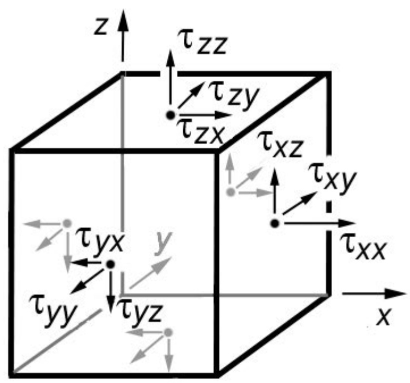

Figure 3.3: Frictional forces on a small control volume as a piece of the local momentum budget.

$$\text{pressure force in } z\text{-direction} \quad = \quad - \, \frac{\partial p}{\partial z} \, \Delta x \Delta y \Delta z \; + \; O(\Delta x \Delta y \Delta z^2).$$

In vectorial form, the pressure force is

$$- \, \vec{\nabla} p \; \Delta x \Delta y \Delta z \; + \; \text{smaller terms},$$

where the vector operator $\vec{\nabla}$, nicknamed 'del', is the gradient operator, the collection $(\partial/\partial x,\, \partial/\partial y,\, \partial/\partial z)$ of first derivatives.

The gravitational force is mg, with m the mass of the fluid parcel and g the earth's gravitational acceleration ($g = 9.81$ m/s^2). The mass of the parcel is its density ρ times its volume $\Delta x \Delta y \Delta z$. Thus,

$$\text{gravitational force} \; = \; \rho g \Delta x \Delta y \Delta z,$$

which acts in the negative z-direction (assuming the conventional choice of coordinate z being directed upward). In vectorial form, it is:

$$\rho \vec{g} \; \Delta x \Delta y \Delta z,$$

where the vector gravity $\vec{g}$ has components $(0,\, 0,\, -g)$.

The frictional forces are somewhat more complicated because, on each of the six sides of the parallelepiped, there is a normal component and two tangential components, together forming a three-dimensional vector (Figure 3.3). For example, on the top side (perpendicular to the z-axis), the stress vector $\vec{\tau}_z$ has the normal component τ_{zz}, and the two tangential components τ_{zx} in the x-direction and τ_{zy} in the y-direction.

The net frictional force acting in the x-direction is thus:

$$\left[\tau_{xx} \left(x + \frac{\Delta x}{2}, y, z \right) \; - \; \tau_{xx} \left(x - \frac{\Delta x}{2}, y, z \right) \right] \Delta y \Delta z$$

$$+ \left[\tau_{yx}\left(x, y + \frac{\Delta y}{2}, z\right) - \tau_{yx}\left(x, y - \frac{\Delta y}{2}, z\right)\right] \Delta x \Delta z$$
$$+ \left[\tau_{zx}\left(x, y, z + \frac{\Delta z}{2}\right) - \tau_{zx}\left(x, y, z - \frac{\Delta z}{2}\right)\right] \Delta x \Delta y.$$

Taylor expansions around the central point (x, y, z) yields:

$$\left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] \Delta x \Delta y \Delta z + \Delta x \Delta y \Delta z \, O(\Delta x, \Delta y, \Delta z),$$

with similar expressions for the other two components of the frictional force. These are complicated expressions (especially given the fact that the stresses are functions of the velocity components), and we shall simply write the frictional forces as $\rho F_x \Delta x \Delta y \Delta z$, $\rho F_y \Delta x \Delta y \Delta z$ and $\rho F_z \Delta x \Delta y \Delta z$, in the respective x-, y- and z-directions of space.

Putting all pieces together, after division by $\Delta x \Delta y \Delta z$ and in the limit of the volume shrinking to a point, the three momentum equations become

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} = -\frac{\partial p}{\partial x} + \rho F_x$$
$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho vu)}{\partial x} + \frac{\partial(\rho vv)}{\partial y} + \frac{\partial(\rho vw)}{\partial z} = -\frac{\partial p}{\partial y} + \rho F_y$$
$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho wu)}{\partial x} + \frac{\partial(\rho wv)}{\partial y} + \frac{\partial(\rho ww)}{\partial z} = -\frac{\partial p}{\partial z} - \rho g + \rho F_z.$$

Some simplifications can be performed. First, the derivatives on the left-hand side can be expanded and a number of terms cancel one another by virtue of (3.4):

$$\rho\left(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho F_x \tag{3.7}$$
$$\rho\left(\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + w\,\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho F_y \tag{3.8}$$
$$\rho\left(\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + w\,\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} - \rho g + \rho F_z. \tag{3.9}$$

Then, since variations in density are small, the density ρ may be replaced everywhere by the constant (reference) density ρ_0, with one major exception. This exception is the gravitational term, through which even a slight variation in density may cause an important buoyancy effect in most environmental applications. The resulting simplification is commonly known as the Boussinesq[1] approximation. The three momentum equations then become:

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\,\frac{\partial p}{\partial x} + F_x \tag{3.10}$$

[1] in honor of Joseph Boussinesq (1842–1929), a French mathematician and hydrodynamicist

$$\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + w\,\frac{\partial v}{\partial z} = -\,\frac{1}{\rho_0}\,\frac{\partial p}{\partial y} + F_y \tag{3.11}$$

$$\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + w\,\frac{\partial w}{\partial z} = -\,\frac{1}{\rho_0}\,\frac{\partial p}{\partial z} - \frac{g\rho}{\rho_0} + F_z, \tag{3.12}$$

or, in vectorial form,

$$\frac{\partial \vec{u}}{\partial t} \;+\; (\vec{u}\cdot\vec{\nabla})\,\vec{u} \;=\; -\;\frac{1}{\rho_0}\vec{\nabla}p \;+\; \frac{\rho\vec{g}}{\rho_0} \;+\; \vec{F}_f, \tag{3.13}$$

where the vector $\vec{F}_f$ gathers the components $(F_x,\,F_y,\,F_z)$.

In cylindrical coordinates (with velocity components u in the radial r-direction v in the azimuthal θ-direction, and w in the vertical z-direction), the preceding equations take the form:

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial r} + \frac{v}{r}\,\frac{\partial u}{\partial \theta} + w\,\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\,\frac{1}{\rho_0}\,\frac{\partial p}{\partial r} + F_r \tag{3.14}$$

$$\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial r} + \frac{v}{r}\,\frac{\partial v}{\partial \theta} + w\,\frac{\partial v}{\partial z} + \frac{uv}{r} = -\,\frac{1}{\rho_0}\,\frac{1}{r}\,\frac{\partial p}{\partial \theta} + F_\theta \tag{3.15}$$

$$\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial r} + \frac{v}{r}\,\frac{\partial w}{\partial \theta} + w\,\frac{\partial w}{\partial z} = -\,\frac{1}{\rho_0}\,\frac{\partial p}{\partial z} - \frac{g\rho}{\rho_0} + F_z. \tag{3.16}$$

We recognize here the so-called centripedal acceleration $(-v^2/r)$ on the left-hand side of the radial-momentum equation.

Example 3.3

A steady wind from a plain blows toward a ridge and enters a wind gap. By so doing, it accelerates from 12 to 40 m/s over a distance of 2 km. How does this horizontal acceleration compare to the vertical gravitational acceleration? And, which term in the corresponding momentum equation balances the acceleration term?

If we take x as the direction of the wind, then the wind speed is $u(x)$. Steadiness removes any time dependence, and we can assume that the cross-wind velocity components, v and w are negligible. Friction is not expected to play a major role in a rapidly adjusting situation and is neglected, too. What is left of the set of momentum equations are the following two terms of (3.10):

$$u\,\frac{\partial u}{\partial x} = -\,\frac{1}{\rho_0}\,\frac{\partial p}{\partial x}\;. \tag{3.17}$$

The acceleration term is the one caused by the change in speed, thus $u\partial u/\partial x$. For the given values, it is equal to

$$\begin{aligned} u\,\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) \;&=\; \frac{(u^2_{\text{downstream}} - u^2_{\text{upstream}})/2}{\text{distance}} \\ &=\; \frac{(40^2 - 12^2)/2}{2000} = 0.364\ \text{m/s}^2, \end{aligned}$$

which is very modest (about 4%) compared to the gravitational acceleration $g = 9.81$ m/s^2.

According to the differential equation, the force accompanying (driving) the acceleration is the pressure force. The pressure gradient is negative:

$$\frac{\partial p}{\partial x} \;=\; -\,\rho_0 u\,\frac{\partial u}{\partial x}$$

telling us that the pressure drops as the air accelerates. Physically, the air accelerates by being pushed from the rear. The pressure drop occurring over the 2 km distance is given by an integration over that distance:

$$\begin{aligned}\Delta p \;&=\; -\rho_0 \int_0^{2000\text{m}} u\;\frac{\partial u}{\partial x}\;dx \\ &=\; -\,\frac{1}{2}\,\rho_0\;[u^2]_0^{2000\text{m}} \\ &=\; -(0.5)(1.225)(40^2 - 12^2) \;=\; -\,892\text{ Pa.}\end{aligned}$$

Such pressure difference is small compared to the standard ground atmospheric pressure of 101,300 Pa.

3.1.4 Conservation of energy

We have seen in Section 2.6 that, for most environmental applications, conservation of energy reduces to a heat equation and, moreover, that the heat content of the fluid is mC_pT. On a per-volume basis, the corresponding 'concentration' c is $\rho C_p T$ and application of budget (3.1) for the small volume of Figure 3.1 provides

$$\frac{\partial}{\partial t}(\rho C_p T) + \frac{\partial}{\partial x}(\rho u C_p T) + \frac{\partial}{\partial x}(\rho v C_p T) + \frac{\partial}{\partial x}(\rho w C_p T) = q,$$

where q is the local heat source per volume and per time. Then, using mass conservation (3.4) and taking the specific heat capacity C_p as constant, we obtain:

$$\frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} + w\,\frac{\partial T}{\partial z} = \frac{q}{\rho C_p}\;.$$

Finally, doing as we did in Section 2.6, we can replace the actual density ρ by the constant (reference) value ρ_0 in the right-hand side:

$$\frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} + w\,\frac{\partial T}{\partial z} = \frac{q}{\rho_0 C_p}\;. \tag{3.18}$$

According to the linearized equation of state given in (2.33), temperature is linearly related to density, and in the end the energy equation for environmental fluids reduces to an equation governing density:

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} = - \frac{\alpha q}{C_p} , \qquad (3.19)$$

where, again, q is the local heat source per volume and unit time.

In the absence of internal heat sources, the term q is zero, and the equation reduces to a statement that individual fluid parcels retain their density along their path:

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} = 0. \qquad (3.20)$$

In cylindrical coordinates, we have:

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \frac{v}{r} \frac{\partial \rho}{\partial \theta} + w \frac{\partial \rho}{\partial z} = 0. \qquad (3.21)$$

Example 3.4: The Bernoulli Equation revisited

We can show that the Bernoulli equation can be derived from the momentum equations in the case of a steady flow of an inviscid fluid. Under these conditions, the equations become:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = - \frac{1}{\rho_0} \frac{\partial p}{\partial x} \qquad (3.22)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = - \frac{1}{\rho_0} \frac{\partial p}{\partial y} \qquad (3.23)$$

$$u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = - \frac{1}{\rho_0} \frac{\partial p}{\partial z} - \frac{g\rho}{\rho_0} . \qquad (3.24)$$

Multiplying the first equation by u, the second by v, the third by w, and adding the results yields:

$$\left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \frac{u^2 + v^2 + w^2}{2} = - \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \frac{p}{\rho_0} - \frac{g\rho w}{\rho_0}$$

Then, using (3.20) and noting that $w = (u\partial/\partial x + v\partial/\partial y + w\partial/\partial z)z$, all terms can be grouped on the left-hand side as

$$\left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \left(\frac{u^2 + v^2 + w^2}{2} + \frac{p}{\rho_0} + \frac{g\rho z}{\rho_0} \right) = 0. \qquad (3.25)$$

We note that the expression $(u^2 + v^2 + w^2)/2 + p/\rho_0 + g\rho z/\rho_0$ follows a simple equation, isomorphic to the density equation (3.20). We recognize here the Bernoulli function defined in (2.30).

3.1.5 Material derivative

The preceding equations, (3.10), (3.11), (3.12) and (3.19), all begin with the same operator:

$$\frac{d}{dt} \;=\; \frac{\partial}{\partial t} \;+\; u\,\frac{\partial}{\partial x} \;+\; v\,\frac{\partial}{\partial y} \;+\; w\,\frac{\partial}{\partial z}\;, \tag{3.26}$$

which is called the *material derivative* or *substantial derivative*. Indeed, if one follows a fluid parcel along its trajectory $[x(t),\; y(t),\; z(t)]$, then any attribute c of this parcel becomes a function of time: $c[x(t), y(t), z(t), t]$, of which the rate of change over time is:

$$\begin{aligned}\frac{dc}{dt} &= \frac{\partial c}{\partial t} + \frac{dx}{dt}\frac{\partial c}{\partial x} + \frac{dy}{dt}\frac{\partial c}{\partial y} + \frac{dz}{dt}\frac{\partial c}{\partial z}\\ &= \frac{\partial c}{\partial t} + u\,\frac{\partial c}{\partial x} + v\,\frac{\partial c}{\partial y} + w\,\frac{\partial c}{\partial z}\;,\end{aligned} \tag{3.27}$$

since, by definition, the velocity components are $u = dx/dt$, $v = dy/dt$ and $w = dz/dt$. Among the four derivatives, the three terms with a spatial derivative and a velocity component as coefficient are called *advective terms*, representing the process known as *advection*, meaning the transport of the quantity by the flow[2].

When such expression is zero, as is the case for the energy equation (3.20) in the absence of heat sources or the Bernoulli equation (3.25), the quantity is conserved by individual fluid parcels.

3.2 Hydrostatic Approximation

Environmental fluids tend to occupy spaces that are far wider than they are thick. For example, the depth of most rivers and lakes is much less than their width, and the lower atmosphere called troposphere extends vertically over about 10 km whereas the weather patterns that it contains stretch over horizontal distances on the order of 1000 km. Such geometrical restriction translates into vertical velocities that are most often much weaker than horizontal velocities. By contrast, gravity is a comparatively strong force acting in the vertical.

This combination of facts renders the acceleration term and the frictional force very small in the vertical momentum equation compared to the gravitational term. As a result, Equation (3.12) can, in good approximation, be reduced to

$$\frac{\partial p}{\partial z} \;=\; -\;\rho g. \tag{3.28}$$

[2]In the past, transport by the flow was called *convection*. The use of this term is now restricted to describe the movement of a fluid caused by temperature differences.

This is the *hydrostatic balance*, which was encountered in the case of a fluid at rest [see (2.21)] and which is seen here to remain valid in the presence of fluid motion, as long as the system is far wider than it is deep.

Note, however, that only the large-scale flow of the fluid, that which senses the disparity between the horizontal and vertical directions, is hydrostatic. The same cannot be said of internal motions at shorter scales, especially the turbulent motions, which are most often not in hydrostatic balance.

3.3 Earth's Rotation

When the horizontal extent of the flow is large and its velocities are small, effects due to the rotation of the earth may become important. In such case, so-called Coriolis[3] terms must be added to the horizontal momentum equations (3.10) and (3.11). A new coefficient appears:

$$f \;=\; 2\Omega\sin(\text{latitude}), \tag{3.29}$$

called the *Coriolis parameter*. Here, Ω is the angular rate of rotation of the earth, equal to $7.29 \times 10^{-5}\ \text{s}^{-1}$.

With the hydrostatic approximation, the momentum equations become:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + F_x \tag{3.30}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + F_y \tag{3.31}$$

$$\frac{\partial p}{\partial z} = -\rho g. \tag{3.32}$$

For a detailed discussion of how the Coriolis terms arise in the momentum equations, the reader is referred to textbooks on *Geophysical Fluid Dynamics* (ex. Cushman-Roisin and Beckers, 2011, Chapter 2).

3.4 Scales and Dimensionless Numbers

3.4.1 Definition of scales

To discern whether a physical process is dynamically important in any particular situation, it is advantageous to introduce *scales of motion*. These are dimensional quantities expressing the overall magnitude of the variables under consideration.

[3]in honor of Gaspard Gustave de Coriolis (1792–1843), a French mechanical engineer

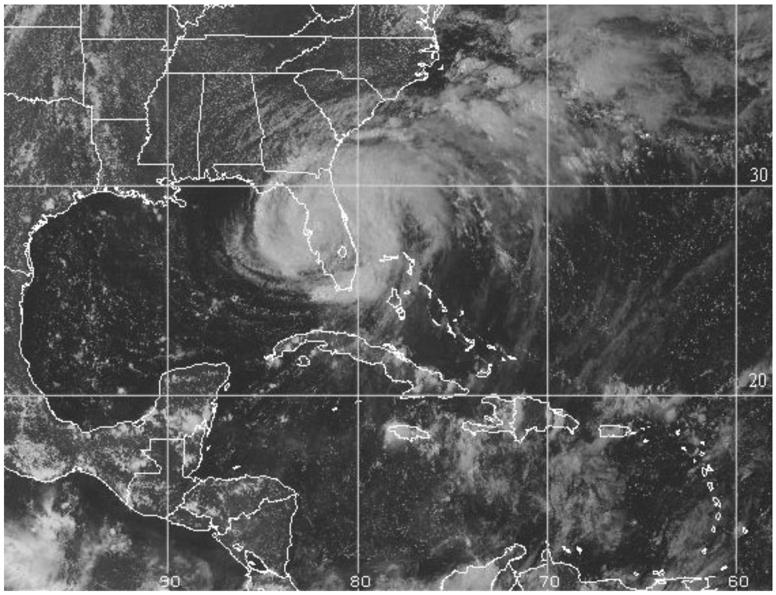

Figure 3.4: Hurricane Frances during her passage over Florida on 5 September 2004. The diameter of the storm is about 830 km and its top wind speed approaches 200 km per hour. (Courtesy of NOAA, Department of Commerce, Washington, D.C.)

They are estimates rather than precisely defined quantities and are understood solely as *orders of magnitude* of physical variables. In most situations, the key scales are those for length (L) and velocity (U). In most environmental applications, the time scale is rather accessory and may simply be taken as the advective time scale L/U, the time it takes the fluid to travel the distance L at speed U.

For example, the wind-driven circulation in a lake can be characterized as follows: since the circulation is primary horizontal, the natural choice of length scale L is the width of the lake basin, while the velocity scale U may be taken as the average of several measurements.

As a second example, consider Hurricane Frances during her course over the southeastern United States in early September 2004 (Figure 3.4). The satellite picture reveals a nearly circular feature spanning approximately 7.5° of latitude (830 km). Sustained surface wind speeds of a category-4 hurricane such as Frances range from 59 to 69 m/s. In general, hurricane tracks display appreciable change in direction and speed of propagation over 2-day intervals. Altogether, these elements suggest the following choice of scales for a hurricane: L = 800 km, U = 60 m/s and time scale = 2 days.

As a third example, consider the discharge of a pollutant in a river. The velocity scale clearly has to be the average velocity of the water (volumetric discharge divided by the channel's cross-sectional area). For the length scale, one has a choice: the depth of the river (if the focus is on vertical mixing or sedimentation), the width of the river (if the focus is on cross-channel mixing) or a certain downstream length (if the focus is on the downstream distance over which the pollutant has some detrimental effect). Likewise, the choice of time scale must reflect the particular choice of physical processes being investigated in the system. Table 1.2 lists a number of possible scales for a variety of environmental processes and systems.

There are three additional scales that play important roles in analyzing environmental fluid problems. As we mentioned earlier, environmental fluids generally exhibit a certain degree of density heterogeneity, called stratification. The important parameters are then the average density ρ_0, the range of density variations $\Delta\rho$, and the height H over which such density variations occur. In the ocean, the weak compressibility of water under changes of pressure and temperature translates into values of $\Delta\rho$ always much less than ρ_0, whereas the compressibility of air renders the selection of $\Delta\rho$ in atmospheric flows somewhat delicate. Since environmental flows are generally bounded in the vertical direction, the total depth of the fluid may be substituted for the height scale H. Usually, the smaller of the two height scales is selected.

As the person new to environmental fluid dynamics has already realized, the selection of scales for any given problem is more an art than a science. Choices are rather subjective. The key is to choose quantities that are relevant to the problem, yet simple to establish. There is freedom. Fortunately, small inaccuracies are inconsequential because the scales are meant only to guide in the clarification of the problem, whereas grossly inappropriate scales will usually lead to flagrant contradictions. Practice, which forms intuition, is necessary to build confidence.

3.4.2 The Froude number

Dimensionless numbers arise in the comparison between terms of the equations governing the motion, and one of particular importance in Environmental Fluid Mechanics is the Froude number. This number arises in water systems (rivers, lakes, estuaries, etc.) because of the free surface.

Water is treated as incompressible (ρ taken equal to ρ_0), and the hydrostatic balance (3.28) provides a scale for the pressure in the fluid. If H is the water depth, then:

$$\frac{\partial p}{\partial z} = -\rho_0 g \qquad \rightarrow \qquad \frac{P}{H} \sim \rho_0 g,$$

where P is the pressure scale and the symbol $\sim$ (meaning 'on the order of') is used to indicate an equivalence in terms of scales. From $P \sim \rho_0 g H$ follows a scale for the pressure force in the momentum equations

$$\frac{1}{\rho_0}\frac{\partial p}{\partial x} \text{ and } \frac{1}{\rho_0}\frac{\partial p}{\partial y} \sim \frac{P}{\rho_0 L} \sim \frac{gH}{L},$$

where L is the horizontal length scale.

We now compare this pressure force to a representative inertial term:

$$u \frac{\partial u}{\partial x} \sim \frac{U^2}{L} ,$$

where U is the horizontal velocity scale. The ratio of inertia to pressure force is:

$$\frac{\text{inertia}}{\text{pressure force}} = \frac{U^2/L}{gH/L} = \frac{U^2}{gH} .$$

The *Froude number*[4] is traditionally defined as the square root of this ratio:

$$Fr = \frac{U}{\sqrt{gH}} . \quad (3.33)$$

The usefulness of the Froude number resides in its ability to categorize flows as slow, intermediate or fast. In the nomenclature of rivers, the following regimes are distinguished (see Chapter 17 for details):

- $Fr < 1$: Flow is fluvial,
- $Fr = 1$: Flow is critical,
- $Fr > 1$: Flow is torrential (or shooting).

Example 3.5: Transition in a small stream

Let us compare the Froude numbers before and after an abrupt change along a small stream (Figure 3.5). Upstream, the water velocity and depth are, respectively, 1.2 m/s and 40 cm. Past an abrupt reduction in channel width of 13%, the velocity has increased, and, by conservation of the Bernoulli function, the water depth has dropped. The new values are 1.9 m/s and 29 cm.

The upstream Froude number is $Fr_1 = (1.2)/\sqrt{9.81 \times 0.40} = 0.61$, which is less than one, while the downstream Froude number is $Fr_2 = (1.9)/\sqrt{9.81 \times 0.29} = 1.13$, which now exceeds one. Thus, in the wider channel upstream, the flow is fluvial, that is, calm and with a smooth surface, whereas in the narrower channel, the flow is torrential, instabilities are manifest, and the surface is rough.

3.4.3 The Richardson number

When the fluid (air or water) is not homogeneous, density differences generate buoyancy forces, which may be responsible for movements of the fluid that would not otherwise exist. The Richardson number[5] compares these buoyancy-related velocities to other velocities that exist in the system. Put another way, it compares buoyancy forces to inertia, or, alternatively, potential energy to kinetic energy.

We start again with the hydrostatic balance (3.28), but this time after having removed the base pressure that stems from the reference density ρ_0, leaving a pressure perturbation p' due exclusively to variations $\rho' = \rho - \rho_0$ of density:

[4] attributed to William Froude (1810–1871), a British naval architect

[5] already encountered in Section 1.3

Figure 3.5: The Sugar River, a small stream in New Hampshire (USA) exhibiting a transition from a smooth surface (in foreground) capable of reflecting the surrounding trees to a rough surface (upper left of photo). The change is caused by an abrupt narrowing cross-section, which forces the stream to accelerate and reduce its water depth. The Froude number switches from being subcritical (< 1) to supercritical (> 1). (Photo by the author)

$$\frac{\partial p'}{\partial z} \;=\; -\,\rho' g \qquad \rightarrow \qquad \frac{\Delta P}{H} \;\sim\; g \Delta\rho,$$

where ΔP and $\Delta\rho$ are the respective scales of p' and ρ', and H the vertical length scale. Solving for ΔP, we obtain:

$$\Delta P \;\sim\; g H \Delta\rho.$$

We now turn to the horizontal momentum equations and compare a typical inertial term

$$u\,\frac{\partial u}{\partial x} \;\sim\; \frac{U^2}{L}$$

to the pressure-gradient force resulting from density variations:

$$\frac{1}{\rho_0}\,\frac{\partial p'}{\partial x} \;\sim\; \frac{\Delta P}{\rho_0 L} \;\sim\; \frac{g H \Delta\rho}{\rho_0 L}\;.$$

The ratio of pressure force to inertial force is called the *Richardson number*:

$$Ri \;=\; \frac{g H \Delta\rho}{\rho_0 U^2}\;. \tag{3.34}$$

In meaning, the Richardson number is very similar to the Froude number, with the chief differences that the Richardson number has the velocity in the denominator and considers buoyancy forces (caused by density differences in the fluid) in contrast to the Froude number, which retains the full force of gravity (due to the absolute density of the fluid).

The Richardson number can also be viewed as an energy ratio, potential energy over kinetic energy. In this perspective, it compares (as it was done in Section 1.3) the supply of kinetic energy to the potential energy held in the stratification. The physical interpretation, therefore, goes as follows:

- If $Ri \ll 1$, stratification is weak and vulnerable to mixing;
- If $Ri \sim 1$, stratification is important and may be reduced by mixing;
- If $Ri \gg 1$, stratification is strong and resisting mixing.

3.4.4 The Reynolds number

This number measures the importance of inertia relative to friction, when friction is caused by viscosity. The component F_x of the frictional force in the x-momentum equations (3.10) was defined as

$$F_x = \frac{1}{\rho}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right).$$

In the case of a simple undirectional shear flow $u(y)$, the active stress component is τ_{xy}, which is related to the velocity shear du/dy by the viscosity μ:

$$\tau_{xy} = \mu \frac{du}{dy} . \tag{3.35}$$

Thus, with length and velocity scales L and U:

$$F_x = \frac{1}{\rho}\frac{d\tau_{xy}}{dy} = \frac{\mu}{\rho}\frac{d^2u}{dy^2} \sim \frac{\mu}{\rho_0}\frac{U}{L^2} ,$$

whereas a representative inertial term scales as

$$u\frac{\partial u}{\partial x} \sim \frac{U^2}{L} .$$

The ratio

$$\frac{\text{inertia}}{\text{viscous force}} = \frac{U^2/L}{\mu U/\rho_0 L^2} = \frac{\rho_0 UL}{\mu}$$

is called the *Reynolds number*[6]:

$$Re = \frac{\rho_0 UL}{\mu} . \tag{3.36}$$

This number is key in distinguishing between laminar and turbulent flows. Laminar flows are spatially smooth and vary slowly over time, while turbulent flows

[6] already mentioned in Section 1.3

exhibit a chaotic behavior marked by ceaseless eddying and temporal fluctuations. The transition from laminar to turbulent regime occurs around a critical Reynolds number, the value of which depends on the geometry of the system (Munson et al., 1994):

- Pipe flow: $Re = 2100$, where L is the pipe diameter,
- Boundary layer: $Re = 5 \times 10^5$, where L is the length of the boundary,
- Flow around an obstacle: $Re = 500$, where L is the width of the obstacle.

Reynolds numbers of environmental flows are invariably very large because of the smallness of the fluid viscosity [$\mu = 1.8 \times 10^{-5}$ kg/(m·s) for air, and 1.0×10^{-3} kg/(m·s) for water] and the large dimensions of outdoor systems. For example, a 2-m wide river flowing at 0.5 m/s is characterized by $Re = 10^6$, and a 1000-m thick diurnal atmospheric boundary layer with a 10 m/s wind by $Re = 7 \times 10^8$.

If the Reynolds number is well above the critical value, its actual value no longer matters. For this reason, Reynolds numbers are rarely mentioned in environmental fluid mechanics.

3.4.5 The Peclet number

A number similar to the Reynolds number is one that measures not the importance of viscosity but the importance of diffusivity. It is called the *Peclet number*[7] and defined as

$$Pe = \frac{UL}{D} , \tag{3.37}$$

where the velocity scale U and length scale L are defined as previously (usually attached to the horizontal direction in the system), and D is the diffusivity of the substance being carried by the fluid. Most often D is taken not as the molecular diffusivity but as the turbulent diffusivity, reflecting the ability of turbulent motions to disperse the substance within the fluid.

The Peclet number compares the relative efficacy of advection and dispersion in transporting the substance across the system. If $Pe \ll 1$, advection is much weaker than diffusion, transfer of the substance from one place in the system to another is mostly effected by turbulent dispersion. Mathematically, the advection term may be neglected in the budget for the substance. In contrast if $Pe \gg 1$, advection is much stronger than diffusion. Spreading is almost inexistent, and the patch of substance is simply moved along by the flow. Mathematically, the diffusion term may be ignored in the budget for the substance. [Note then that the neglect of the term with the highest-order derivative reduces the need of boundary conditions by one. No boundary condition may be imposed at the downstream end of the domain: what happens there is whatever the flow brings from the inside.] Finally, if $Pe \sim 1$, advection and diffusion are equally important, and neither may be ignored. No reduction of the equation can be justified, and the full budget equation must be utilized.

[7] in honor of Jean Claude Eugène Péclet (1793–1857), a French physicist

3.4.6 The Rossby number

Another dimensionless number arises when the Coriolis terms are retained in the momentum equations to take into account the effects of the earth's rotation. According to (3.30) and (3.31), the additional terms are $-fv$ and $+fu$, respectively, where $f = 2\Omega \sin(\text{latitude})$. Comparing the scale of these new terms

$$-fv \;\text{ and }\; +fu \;\sim \Omega U$$

once again to inertia ($\sim U^2/L$), the ratio

$$\frac{\text{inertia}}{\text{Coriolis}} \;=\; \frac{U^2/L}{\Omega U} \;=\; \frac{U}{\Omega L}$$

is called the *Rossby number*[8]:

$$Ro \;=\; \frac{U}{\Omega L} \;. \tag{3.38}$$

where, we recall, the earth's rotation rate Ω is $7.29 \times 10^{-5}\ \text{s}^{-1}$.

The meaning of the Rossby number is as follows: If $Ro \ll 1$ or $Ro \sim 1$, the earth's rotation has important effects and must be retained in the formulation of the problem, but if $Ro \gg 1$, it is not and can be ignored. Most environmental systems fall in this latter category and the Coriolis terms do not need to be included in the momentum equations. It is only at large, geophysical scales, such as weather patterns and ocean currents, that they need to be retained. For example, Hurricane Frances (Figure 3.4) is characterized, as mentioned earlier, by $U = 60$ m/s, $L = 800$ km and therefore $Ro = 1.0$. Weather patterns, which are larger and have weaker winds, have Rossby numbers that fall significantly below one.

3.5 Vorticity

The generally turbulent character of environmental flows reveals much swirling and eddying. These somewhat circular motions are crucial in mixing the fluid and diluting pollutants. But, why do they occur? The reason lies in the fact that vorticity, i.e. the amount of 'spin' in the fluid, has a peculiar behavior. It is therefore useful to study vorticity in the context of environmental flows.

Vorticity is defined as the curl of the velocity vector. It is itself a vector:

$$\vec{\omega} \;=\; \vec{\nabla} \times \vec{u}, \tag{3.39}$$

or, component by component:

$$\omega_x \;=\; \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \;, \quad \omega_y \;=\; \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \;, \quad \omega_z \;=\; \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \;. \tag{3.40}$$

[8]in honor of Carl-Gustav Rossby (1898–1957), a Swedish meteorologist

Its temporal evolution is deduced from that of momentum. To show this, we take the curl ($\vec{\nabla}\times$) of the vectorial momentum equation (3.13). A significant amount of algebraic manipulations, including some vectorial calculus identities and the use of continuity equation (3.5), helps reduce the equation to:

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\ \vec{\omega} - (\vec{\omega}\cdot\vec{\nabla})\ \vec{u} \ = \ -\ \frac{1}{\rho_0}\ \vec{\nabla}\times\vec{\nabla}p + \vec{b} + \vec{\nabla}\times\vec{F}_f.$$

Several things happen here. First and foremost, because the curl of a gradient is always zero, the pressure term (first on the right-hand side) vanishes. Physically, this is due to the fact that pressure, a normal force, is incapable of providing a spin to a fluid parcel. It implies that the only two sources of vorticity are buoyancy and friction. The buoyancy term $\vec{b}$ has the following components:

$$b_x = -\ \frac{g}{\rho_0}\ \frac{\partial \rho}{\partial y}\ , \qquad b_y = +\ \frac{g}{\rho_0}\ \frac{\partial \rho}{\partial x}\ , \qquad b_z = 0, \tag{3.41}$$

in which we note, somewhat surprisingly, that the vertical buoyancy force induces vorticity only in the horizontal directions. We shall return to this 'twist'. Finally, because it includes tangential forces, friction acts as a source of vorticity.

In the absence of the frictional force (the interesting case) and in terms of the individual vorticity components, the equations are:

$$\frac{d\omega_x}{dt} \ = \ \omega_x\ \frac{\partial u}{\partial x}\ +\ \omega_y\ \frac{\partial u}{\partial y}\ +\ \omega_z\ \frac{\partial u}{\partial z}\ -\ \frac{g}{\rho_0}\ \frac{\partial \rho}{\partial y} \tag{3.42}$$

$$\frac{d\omega_y}{dt} \ = \ \omega_x\ \frac{\partial v}{\partial x}\ +\ \omega_y\ \frac{\partial v}{\partial y}\ +\ \omega_z\ \frac{\partial v}{\partial z}\ +\ \frac{g}{\rho_0}\ \frac{\partial \rho}{\partial x} \tag{3.43}$$

$$\frac{d\omega_z}{dt} \ = \ \omega_x\ \frac{\partial w}{\partial x}\ +\ \omega_y\ \frac{\partial w}{\partial y}\ +\ \omega_z\ \frac{\partial w}{\partial z}\ . \tag{3.44}$$

When the fluid has a uniform density (ρ derivatives being zero), we note that every term in the equations contains a vorticity component and, therefore, that vorticity can only change because of vorticity itself. Indeed, if the flow were deprived of vorticity at some original time ($\vec{\omega} = 0$ at $t = 0$), then its vorticity would remain zero at all subsequent times. This, in effect, says that a homogeneous fluid redistributes vorticity but does not create it. The precise mathematical statement corresponding to this property, called the Kelvin circulation theorem, will be established in the next section.

By contrast, a fluid with variable density, such as a stratified fluid, has the ability to generate its own vorticity as soon as its density varies in the horizontal ($\partial\rho/\partial x$ and/or $\partial\rho/\partial y$ nonzero). If initially the fluid is perfectly stratified in the vertical (ρ function of z only), all it takes is some divergence in the horizontal flow ($\partial u/\partial x + \partial v/\partial y \neq 0$) to generate some vertical velocity ($\partial w/\partial z \neq 0$) which in turn will cause some horizontal variation of density via the density equation (3.20). The amount of vorticity production is the subject of the Bjerknes circulation theorem and will be determined in the next section.

When the hydrostatic balance is invoked, the vertical momentum equation is drastically reduced and the vorticity equations undergo some reductions. These reductions are equivalent to keeping the same equations but redefining the vorticity components as:

$$\omega_x = -\frac{\partial v}{\partial z}, \quad \omega_y = \frac{\partial u}{\partial z}, \quad \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{3.45}$$

which amounts to dropping the vertical velocity terms next to their larger horizontal counterparts, in keeping with the premise that vertical motions are weak in systems that are far wider than they are deep.

With Coriolis terms present in the formulation, the vorticity equations remain the same, with one adaptation: The Coriolis parameter is added to the vertical vorticity component ω_z:

$$\omega_z \quad \rightarrow \quad \omega_z + f. \tag{3.46}$$

Physically, this means that the vertical vorticity component is augmented by vorticity due to the ambient rotation.

Example 3.6: Vorticity in a river bend

In a river, water flows faster at the surface than near the bottom, where it is retarded by friction. If x is the downstream direction and z is upward, then the velocity is $u(z)$, and the shear du/dz is positive. This shear contributes to a cross-stream vorticity ω_y, directed to the left of the current (Figure 3.6).

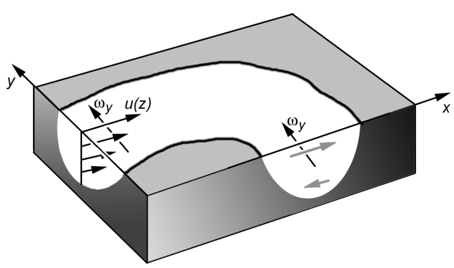

Figure 3.6: Vorticity dynamics in a river bend. The vorticity ω_y induced by bottom friction before the bend manifests itself as a cross-stream circulation after the bend.

In a river bend, vorticity adapts but this y-component persists for a while, imparting the fluid with a vorticity component aligned with the flow. The manifestation of this vorticity is a cross-stream secondary circulation with flow toward the outer bank on the surface and counterflow along the bottom (Figure 3.6).

3.6 Circulation Theorems

Circulation is an alternative quantity that measures the amount of spin in a flow. It is defined as the integral of the tangential component of velocity along a closed curve (loop), which, by virtue of Stokes' theorem, is equal to the flux of vorticity through the enclosed area (Figure 3.7):

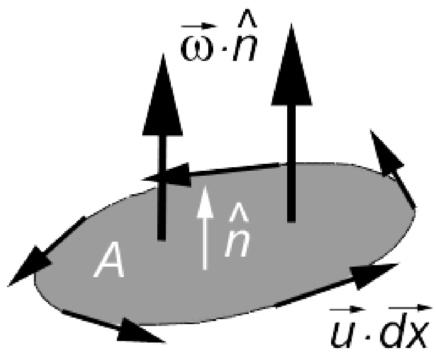

Figure 3.7: Circulation is defined as the integral of the tangential component of velocity along a loop. It is also equal to the flux of vorticity through the area encompassed by the loop.

$$\Gamma = \oint \vec{u} \cdot d\vec{x} = \iint \vec{\omega} \cdot \hat{n} \, dA. \tag{3.47}$$

In other words, circulation is some form of aggregate vorticity.

Consider now two fluid parcels at positions $\vec{x}_1$ and $\vec{x}_2$ that are so close to each other that the distance between them, $d\vec{x} = \vec{x}_2 - \vec{x}_1$, is a small differential. Let us follow these two points as they move with the fluid. The material time derivative of the positions are the respective velocities:

$$\frac{d\vec{x}_1}{dt} = \vec{u}_1 \, , \quad \frac{d\vec{x}_2}{dt} = \vec{u}_2$$

and their difference is the velocity differential:

$$\frac{d}{dt} d\vec{x} = \vec{u}_2 - \vec{u}_1 = d\vec{u}.$$

[The two points share the same d/dt derivative because they are so close to each other.]

Now, take the material derivative of the scalar product $\vec{u} \cdot d\vec{x}$:

$$\frac{d}{dt}(\vec{u} \cdot d\vec{x}) = \frac{d\vec{u}}{dt} \cdot d\vec{x} + \vec{u} \cdot d\vec{u}.$$

Using the vectorial momentum equation (3.13) to replace the acceleration $d\vec{u}/dt$ provides:

$$\frac{d}{dt}(\vec{u} \cdot d\vec{x}) = - \frac{1}{\rho_0} \vec{\nabla} p \cdot d\vec{x} + \frac{\rho}{\rho_0} \vec{g} \cdot d\vec{x} + \vec{F}_f \cdot d\vec{x} + \vec{u} \cdot d\vec{u}.$$

The scalar product $\vec{\nabla}p \cdot d\vec{x} = (\partial p/\partial x)dx + (\partial p/\partial y)dy + (\partial p/\partial z)dz$ is equal to the total differential $dp = p(\vec{x}_2) - p(\vec{x}_1)$. The next scalar product is simple: $\vec{g} \cdot d\vec{x} = -gdz$. Finally, the last term can be rewritten as $\vec{u} \cdot d\vec{u} = d[(\vec{u} \cdot \vec{u})/2] = d[(\vec{u}_2 \cdot \vec{u}_2) - (\vec{u}_1 \cdot \vec{u}_1)]/2$.

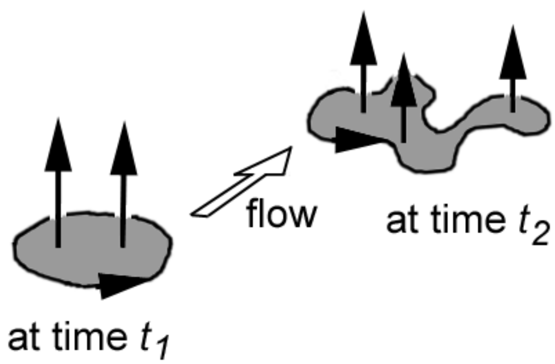

Figure 3.8: Visualization of the Kelvin circulation theorem. A loop moving with the fluid may become distorted but the flux of vorticity through it remains unchanged if the fluid is homogneous and frictionless. If the loop area area shrinks, the vorticity increases proportionally; if it expands, the vorticity diminishes.

Now, we add many of these small distances to complete a loop as shown on Figure 3.7 and obtain:

$$\frac{d}{dt} \oint \vec{u} \cdot d\vec{x} \; = \; - \; \frac{1}{\rho_0} \oint dp \; - \; \frac{g}{\rho_0} \oint \rho dz \; + \; \oint \vec{F}_f \cdot d\vec{x} \; + \; \frac{1}{2} \oint d(\vec{u} \cdot \vec{u}).$$

Since on a closed loop the start and end values of p and $(\vec{u} \cdot \vec{u})/2$ are the same, the respective integrals vanish, and we have:

$$\frac{d\Gamma}{dt} \; = \; - \; \frac{g}{\rho_0} \oint \rho dz \; + \; \oint \vec{F}_f \cdot d\vec{x}, \tag{3.48}$$

which shows that the only two agents capable of generating new circulation in the fluid are density variations and friction.

In the absence of density variations ($\rho = \rho_0$) and of friction ($\vec{F}_f = 0$), circulation is conserved:

$$\frac{d\Gamma}{dt} = 0. \tag{3.49}$$

This is the Kelvin[9] circulation theorem, which can be stated in words as: *The circulation around a closed curve moving with the fluid remains constant with time.* In other words, circulation is a constant of the motion.

[9]Famous Scottish mathematician, scientist and inventor (1824–1907), born under the name of William Thomson and ennobled late in life. The absolute temperature scale is named after him.

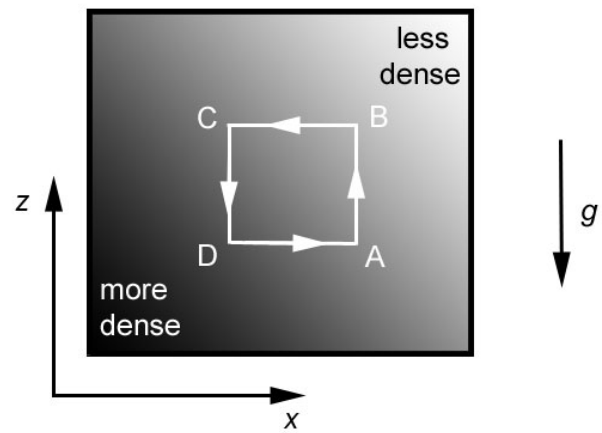

Figure 3.9: Physical interpretation of the Bjerknes circulation theorem. When the density gradient has a horizontal component, the theorem states that the circulation around a loop (such as A-B-C-D-A) will be increasing or decreasing. It is clear physically that lighter fluid has a tendency to rise and float on top while the denser fluid has a tendency to sink and spread on the bottom. Motion is therefore induced in the direction of the arrows of the loop. The circulation predicted to arise by the Bjerknes theorem is thus none other than the slumping of the fluid as it seeks a lower-energy level under the influence of gravity.

Since vorticity and circulation are intertwined [see definition (3.47), preservation of circulation implies that the vorticity flux through a loop remains constant as that loop is carried by the fluid (Figure 3.8).

Environmental fluids often exhibit density variations, and the Kelvin circulation theorem is not applicable. It is worth considering then how much circulation can be generated by buoyancy forces. Restoring the density term in the right-hand side of (3.48) provides:

$$\frac{d\Gamma}{dt} = - \frac{g}{\rho_0} \oint \rho \, dz, \tag{3.50}$$

This generalization is known as the Bjerknes[10] circulation theorem. Its physical interpretation is quite straightforward.

Consider a stratified fluid with a density distribution that has a horizontal component (Figure 3.9). Naturally, the lighter fluid has a tendency to float to the higher levels while the denser fluid has a tendency to sink to the bottom. This generates a convective circulation. In a vertical plane inside this fluid, draw a rectangular loop A→B→C→D→A (Figure 3.9). Along the ascending branch A→B, $dz > 0$ and the

[10] Vilhelm Bjerknes (1862–1951), Norwegian meteorologist who set meteorology on a physical basis

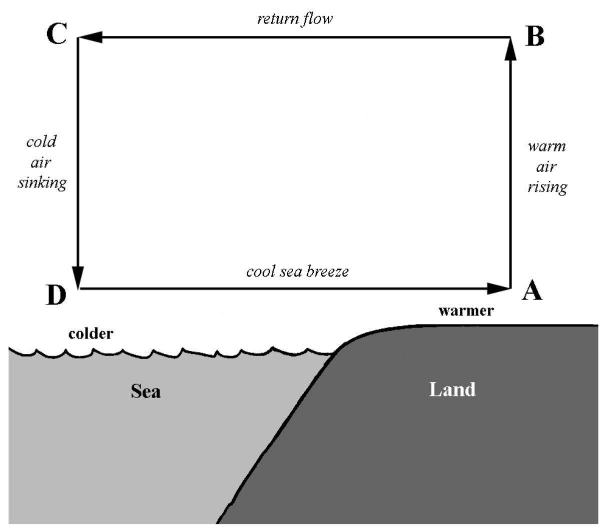

Figure 3.10: Application of the Bjerknes circulation theorem to the sea breeze. Air sinks over the colder sea, blows toward the land, where it is heated and rises. Positive circulation ensues around a vertical rectangular loop spanning the coastline.

contribution $\int_A^B \rho dz$ to the overall integral is positive. Along the descending branch C→D, $dz < 0$ but density is now greater and the integral contribution $\int_C^D \rho dz$ is negative with a larger magnitude. The horizontal segments B→C and D→A involve no change in height ($dz = 0$) and do not contribute to the integral. The net result is a negative integral $\oint \rho dz$, which according to the Bjerknes circulation theorem (3.50) creates an increase in circulation Γ around the loop. Positive circulation means flow in the direction of the tracing of the loop, which is exactly what is expected when the lighter fluid moves to occupy the top, and the denser fluid sinks to the bottom.

Example 3.7: Sea breeze

Because water is relatively transparent but land opaque, the solar radiation penetrates to greater depth in the sea than in the ground by the shore. In addition, the heat capacity of water is greater than that of soil, implying that for equal heat input, the temperature of the sea increases less than that of land. During daytime by the seashore, these differences generally translate into a warmer land next to a colder sea. Warm air rises over the land, drawing cooler air from the sea (Figure 3.10).

Consider a vertical rectangular loop consisting in two horizontal segments spanning the coastline and in two vertical segments, one over the sea and the other over the land, as drawn in Figure 3.10. Since the horizontal segments (where $dz = 0$) do not contribute to the loop integral, only the vertical components do, and the Bjerknes circulation theorem (3.50) reduces to:

$$\frac{d\Gamma}{dt} \;=\; -\;\frac{g}{\rho_0}\int_A^B \rho\, dz \;-\; \frac{g}{\rho_0}\int_C^D \rho\, dz.$$

With $\rho = \rho_0[1 - \alpha(T - T_0)]$, this becomes

$$\frac{d\Gamma}{dt} \;=\; +\;\alpha g \int_0^H \;[T(\text{along A–B}) \;-\; T(\text{along C–D})]\; dz,$$

which can be approximated to

$$\frac{d\Gamma}{dt} \;=\; \alpha g H\; (T_{AB} \;-\; T_{CD}), \tag{3.51}$$

where H is the height of the loop, and T_{AB}, T_{CD} the averaged temperatures on the vertical segments A$\rightarrow$B and C$\rightarrow$D. The difference $T_{AB} - T_{CD}$ is nearly the temperature contrast between land and sea.

The result is a positive circulation along the loop, indicative of a wind from sea to land along the surface and a return flow from land to sea aloft. For a loop of length L (say, 20 km) and height H (say 1 km), the average velocity U is $\Gamma/2(L + H) \simeq \Gamma/2L$, and the acceleration is

$$\frac{dU}{dt} \;\simeq\; \frac{\alpha g H}{2L}\;(T_{\text{land}} \;-\; T_{\text{sea}}).$$

For a temperature contrast ($T_{\text{land}} \;-\; T_{\text{sea}}$) of 5°C, the acceleration is estimated to 0.0043 m/s^2, which generates a wind of 15 m/s in one hour. As the wind increases, friction becomes important and the thermal contrast between land and sea diminishes, leading ultimately to a steady state until the sun sets and the breeze dies away.

Problems

3-1. Consider the chemically reacting pollutant mentioned in Example 3.1 and modeled by Equation (3.2) and suppose that the initial concentration distribution consists in a single peak, given by

$$c_0(x) \;=\; \frac{Ca^2}{x^2 + a^2}\;,$$

in which $C = 13$ mg/L and $a = 400$ m. The fluid velocity is $u = 0.5$ m/s and the decay coefficient $K = 0.20$/hour. If the maximum legal concentration according to existing regulations is 0.5 mg/L, determine the duration (in hours) of the pollution episode and the area affected along the x-axis (in kilometers, on both upstream and downstream sides from the origin).

3-2. A river with cross-sectional area of 160 m^2 carries a volumetric discharge of 23 m^3/s. At some location $x = 0$ along it, 1 m^3/s of partially treated sewage is discharged. This sewage contains a biochemical oxygen demand (BOD) concentration of 30 mg/L, which decays in water at the rate $K = 0.33$ day^{-1}. Trace the BOD concentration downstream from the point of discharge. In particular, what is the BOD concentration immediately after mixing in the river water (assuming no appreciable decay in this rapid phase)? And, how far downstream has the BOD concentration fallen below 0.5 mg/L? Assume steady state.

3-3. The horizontal circulation in a lake in the vicinity of the coast (straight line $x = 0$) can be approximated by the following functions

$$\begin{aligned} u(x,y) &= -\,ax \\ v(x,y) &= +\,ay, \end{aligned}$$

where a is a positive constant. What is the structure of this circulation and what can you say about the vertical velocity? Assume a flat horizontal bottom at depth H.

3-4. If a fluid moves in concentric circles, but not necessarily as a solid body, with the following velocity distribution

$$u = -\,\Omega(r)\;y \qquad v = +\,\Omega(r)\;x \qquad w = 0,$$

where r is the radial distance ($r^2 = x^2 + y^2$) and $\Omega(r)$ is a differentiable function of r, find the acceleration of every fluid particle as a function of its position (x, y). What is its direction (radial, azimuthal or oblique)? Can you explain the direction of the acceleration vector on physical grounds?

3-5. The vortex motion of the preceding problem is a steady solution of the equations with the earth rotation [Equations (3.30)–(3.31)–(3.32)]. Taking f and ρ as constants and neglecting friction, determine the accompanying pressure distribution $p(x, y, z)$ when

$$\Omega(r) = \frac{A}{r^2 + a^2}\;. \qquad (3.52)$$

3-6. Show that in a steady flow of an incompressible and frictionless fluid, the Bernoulli function is not only constant along streamlines but also along vortex lines. (A vortex line is a line everywhere tangent to the local vorticity vector.)

3-7. At some point along its path, a small river has a 12-m wide channel, and, at the time when measurements are taken, its water depth is 0.35 m and its velocity 1.2 m/s. Downstream of this point, the width of the channel gradually decreases. At which channel width does the flow switch from being fluvial to being torrential?

3-8. If the flow speed of a 7.8-m thick layer of surface water is 0.85 m/s while the layer below is at rest, how large should the relative density difference $\Delta\rho/\rho_0$ be to prevent mixing between layers?

3-9. In a slow flow with large density variations, the Richardson number can be very small ($Ri \gg 1$). Show that in such case, the Bernoulli equation can be approximated to

$$p \;+\; \rho g z \;=\; \text{constant}.$$

Then, take the z-derivative of this expression, keeping in mind that density is variable ($\partial\rho/\partial z \neq 0$), and use the hydrostatic balance (3.28). What do you obtain? Resolve any paradox.

3-10. Show that for a two-dimensional flow

$$u \;=\; u(x,y,t) \qquad v \;=\; v(x,y,t) \qquad w \;=\; 0,$$

the Kelvin circulation theorem implies that the vertical component ω_z of vorticity is conserved along the trajectory of every fluid particle.

3-11. Estimate the horizontal circulation and the vertical component of vorticity in Hurricane Fances (see Figure 3.4 and characteristic values in accompanying text).

3-12. Select a hurricane of the recent past and obtain for it the peak wind velocity and diameter on the same day. Estimate the Rossby number. What does this value imply in regards to the importance of the Earth's rotation?

3-13. Suppose that in a shallow water flow, friction can be modeled as a simple retarding force proportional to the velocity:

$$\vec{F}_f \;=\; -\; A\,\vec{u}$$

where A is a positive constant. Derive the corresponding circulation theorem.

3-14. In a certain airshed, the density of the air can be approximated by a bilinear function of x and z according to

$$\rho(x,z) \;=\; \rho_0\,(1 - ax - bz), \tag{3.53}$$

where $\rho_0 = 1.22$ kg/m^3, $a = 1.7 \times 10^{-6}$ m^{-1} and $b = 2.8 \times 10^{-4}$ m^{-1}. According to the Bjerknes circulation theorem, such distribution of density should generate circulation in the (x, z) plane. By how much does circulation increase in the span of 1 hour around a rectangle 10 km long and 500 m high? What is the change in average velocity along this loop during that hour?