Chapter 5

Instabilities

SUMMARY: There are situations when a wave is capable of extracting energy from the system. It does so by drawing either kinetic energy from pre-existing motion or potential energy from background stratification. In either case the wave amplitude grows over time, and the wave is said to be unstable. This chapter reviews the most common types of instability in environmental fluid systems, progressing from smaller billows to weather systems. Instabilities are a prelude to mixing and turbulence.

5.1 Kelvin-Helmholtz Instability

For the sake of simplicity, let us first consider an extreme type of stratification, namely a two-layer system (Figure 5.1) in which a lighter layer of fluid floats over another, heavier layer (such as in a lake in summer). Physical principles tell us that gravity waves can propagate on the interface separating these two layers, not unlike waves propagating on the surface of a pond.

But, if the layers flow at different velocities[1], i.e. when a *shear* is present, these waves may grow in time and lead to overturning in the vicinity of the interface. These breaking internal waves, called *billows*, generate mixing over a height a little shorter than their wavelength (Figure 5.2). The phenomenon is known as the *Kelvin-Helmholtz instability.*

5.1.1 Theory

Because the billows are elongated in the transverse direction (at least initially), the dynamics may be restricted the vertical plane, and only two velocity components need be retained, u in the horizontal and w in the vertical. In the absence of friction,

[1] We consider here the simple case of two distinct but uniform velocities. The case of velocities varying with height has been treated by Redekopp (2002, Section 3.1).

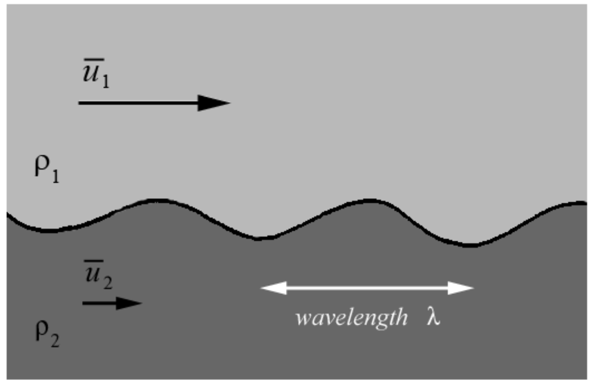

Figure 5.1: A two-layer stratification with shear flow. This configuration is conducive to billow development and vertical mixing.

which is not important until the advanced turbulent stage of the system, and with the Boussinesq approximation, the equations of motion are:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{5.1}$$

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + w\,\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\,\frac{\partial p}{\partial x} \tag{5.2}$$

$$\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + w\,\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\,\frac{\partial p}{\partial z} - \frac{\rho g}{\rho_0}\,. \tag{5.3}$$

It is understood that there are actually two such sets of equations, one for each layer. In the upper layer, the density is ρ_1 and, in the lower layer, ρ_2. The initial stratification has the lighter fluid floating on top of the denser fluid and thus $\rho_1 < \rho_2$.

We decompose the variables in sums of a basic flow plus a wave perturbation. The pressure also includes a hydrostatic component:

$$u = \bar{u} + u' \tag{5.4}$$

$$w = \quad + w' \tag{5.5}$$

$$p = p_0 - \rho g z + p' \tag{5.6}$$

$$\tag{5.7}$$

where $\bar{u}$ takes on the value $\bar{u}_1$ or $\bar{u}_2$, the existing velocities in the upper and lower layers, respectively.

As for wave dynamics, we consider only small-amplitude perturbations so that we may linearize the equations for the perturbations. These are:

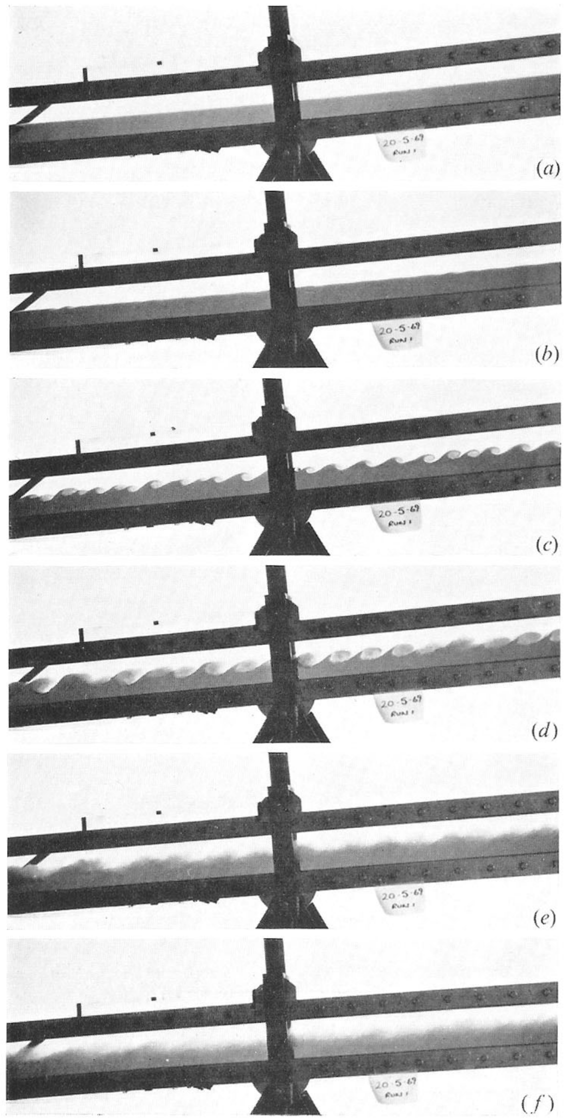

Figure 5.2: Laboratory demonstration of a Kelvin-Helmholtz instability framed as an initial-value problem. At some initial time, a long tank filled with two resting fluids of different densities is tipped slightly. As the denser fluid sinks toward the bottom and the lighter fluid rises toward the top, a counterflow is generated, and the interface develops unstable billows. [From Thorpe, 1971]

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \tag{5.8}$$

$$\frac{\partial u'}{\partial t} + \bar{u}\,\frac{\partial u'}{\partial x} = -\frac{1}{\rho_0}\,\frac{\partial p'}{\partial x} \tag{5.9}$$

$$\frac{\partial w'}{\partial t} + \bar{u}\,\frac{\partial w'}{\partial x} = -\frac{1}{\rho_0}\,\frac{\partial p'}{\partial z}\ . \tag{5.10}$$

Because they are linear, these equations admit solutions in the form of trigonometric functions. However, since we anticipate amplitude growth in addition to phase propagation, we seek a solution of the type

$$\begin{aligned} u' &= U(z)\ \mathrm{e}^{i(kx-\omega t)} \\ w' &= W(z)\ \mathrm{e}^{i(kx-\omega t)} \\ p' &= P(z)\ \mathrm{e}^{i(kx-\omega t)} \end{aligned}$$

of which we retain only the real parts. The wavenumber k corresponds to an actual wavelength $\lambda = 2\pi/k$ and is a real, positive quantity. In contrast, the frequency ω, which needs to be determined in terms of k, may be a complex number. Should it be so, i.e.

$$\omega = \omega_r + i\ \omega_i, \tag{5.11}$$

the real part corresponds to the actual frequency (from which the period can be determined, $T = 2\pi/\omega_r$) whereas the imaginary part leads to a coefficient of the type

$$\mathrm{e}^{+\omega_i t}$$

which grows exponentially in time if ω_i is positive. Thus, the existence of a positive imaginary part in ω is the symptom of unbounded growth and instability.

The equations for the vertical profiles $U(z)$, $W(z)$, $P(z)$ of velocity and pressure are obtained by substitution of the wave forms in Equations (5.8)–(5.9)–(5.10):

$$ikU + \frac{dW}{dz} = 0 \tag{5.12}$$

$$i(k\bar{u} - \omega)\ U = -\frac{ik}{\rho_0}\ P \tag{5.13}$$

$$i(k\bar{u} - \omega)\ W = -\frac{1}{\rho_0}\ \frac{dP}{dz}\ . \tag{5.14}$$

Elimination of U and P using (5.12) and (5.13) yields a single equation for W, which reduces to:

$$\frac{d^2W}{dz^2} = k^2 W, \tag{5.15}$$

of which the general solution consists in a linear combination of the exponentials $\exp(+kz)$ and $\exp(-kz)$. In each layer, we reject the exponentially growing solution and retain only the component decaying away from the interface.

If we introduce the displacement $a(x,t)$ of the interface and ascribe to it a similar wave form:

$$a \;=\; A\ \mathrm{e}^{i(kx-\omega t)}\ , \tag{5.16}$$

then matching the vertical velocity w with this interfacial displacement ($w = da/dt \to w' = \partial a/\partial t + \bar{u}\partial a/\partial x$) gives:

$$\text{Upper layer:} \qquad W_1 \;=\; i(k\bar{u}_1 - \omega)A\ \mathrm{e}^{-kz} \tag{5.17}$$

$$\text{Lower layer:} \qquad W_2 \;=\; i(k\bar{u}_2 - \omega)A\ \mathrm{e}^{+kz} \tag{5.18}$$

The perturbation pressures in both layers are then obtained from (5.14):

$$\text{Upper layer:} \qquad P_1 \;=\; -\ \frac{\rho_0(k\bar{u}_1 - \omega)^2}{k}\ A\ \mathrm{e}^{-kz} \tag{5.19}$$

$$\text{Lower layer:} \qquad P_2 \;=\; +\ \frac{\rho_0(k\bar{u}_2 - \omega)^2}{k}\ A\ \mathrm{e}^{+kz} \tag{5.20}$$

Matching the total pressures (hydrostatic plus perturbation parts) at the interface $z = a$ yields, after some algebra:

$$\frac{\Delta\rho}{\rho_0}\ gk \;=\; (k\bar{u}_1 - \omega)^2 \;+\; (k\bar{u}_2 - \omega)^2 \tag{5.21}$$

in which $\Delta\rho$ stands for the density difference between the two layers:

$$\Delta\rho \;=\; \rho_2 - \rho_1. \tag{5.22}$$

This density difference must be positive, because the initial stratification is gravitationally stable. It is also assumed to be much less than ρ_0 to justify the initial Boussinesq approximation.

The quadratic equation (5.21) for ω is the dispersion relation of the system. As it is solved for ω,

$$\omega \;=\; \frac{1}{2}\left[k(\bar{u}_1 + \bar{u}_2) \pm \sqrt{2gk\frac{\Delta\rho}{\rho_0} - k^2(\bar{u}_1 - \bar{u}_2)^2}\ \right]\ , \tag{5.23}$$

a square root arises. If the radical is negative, the solutions are complex, with one of the possible imaginary parts certain to be positive, and the wave perturbation grows over time. This occurs for wavenumbers k meeting the inequality

$$2g\ \frac{\Delta\rho}{\rho_0} \;<\; k\Delta\bar{u}^2, \tag{5.24}$$

in which $\Delta\bar{u}$ stands for the velocity shear between the two layers:

$$\Delta\bar{u} \;=\; |\bar{u}_1 - \bar{u}_2|. \tag{5.25}$$

Note that if this shear is zero ($\bar{u}_1 = \bar{u}_2$), inequality (5.24) can never be met, the radical is positive for all wavenumbers, and the frequency ω is real. Thus, it is the existence of the shear that enables the instability.

All waves with wavelengths λ satisfying the inequality

$$\lambda \;<\; \frac{\pi\rho_0\Delta\bar{u}^2}{g\Delta\rho}\;, \tag{5.26}$$

are unstable. In other words, all short waves up to a critical wavelength grow in time and turn into billows. The critical wavelength is

$$\lambda_{\text{crit}} \;=\; \frac{\pi\rho_0\Delta\bar{u}^2}{g\Delta\rho} \;=\; \frac{\pi\Delta\bar{u}^2}{\alpha g\Delta T}\;, \tag{5.27}$$

where $\Delta T = T_1 - T_2$ is the temperature difference between the two layers that causes the stratification.

Only waves with wavelength shorter than λ_{crit} grow into billows, and it can be shown that the shorter the wave, the faster is its temporal growth (Problem 5-1). At short wavelengths, however, the damping effect of friction becomes important, and the wavelength of the fastest growing wave happens to be very sensitive to both viscosity and details of the initial perturbation. As laboratory experiments show (Thorpe, 1971) and theory confirms (Hazel, 1972), the billows generally exhibit a wavelength about 12 to 15 times the thickness[2] of the density interface prior to the onset of instability.

The presence of a boundary, at either top or bottom, can further destabilize a stratified shear flow (Hazel, 1972). Lindzen and Rosenthal (1976) explained this by observing that the essence of the instability is an over-reflection of an internal gravity wave at the interface and that, when a boundary is present, a wave over-reflected at the interface can travel to the boundary, be reflected there, and return to the interface for additional amplification. In other words, the presence of a boundary generates an echo between boundary and zone of instability.

Instability was considered here as an initial-value problem. Alternatively, it can be viewed as a boundary-value problem (Figure 5.3), and the theory must be modified accordingly (Cushman-Roisin, 2005). The critical wavelength is found to be longer:

$$\lambda_{\text{crit}} \;=\; \frac{2\pi\rho_0\Delta\bar{u}^2}{g\Delta\rho}\left[1 \;+\; \frac{\bar{u}_1 + \bar{u}_2}{\sqrt{2(\bar{u}_1^2 + \bar{u}_2^2)}}\right]^{-1}\;. \tag{5.28}$$

A common environmental example of the K-H instability as a boundary-value problem is a river entering a colder lake or the salty ocean. Seim and Gregg (1994) hypothesized that their observations in the waters of the Admiralty Inlet (a tidal channel in the vicinity of Seattle, USA) may have been the result of a spatially growing instability.

[2]The interfacial thickness is always finite because of molecular diffusion.

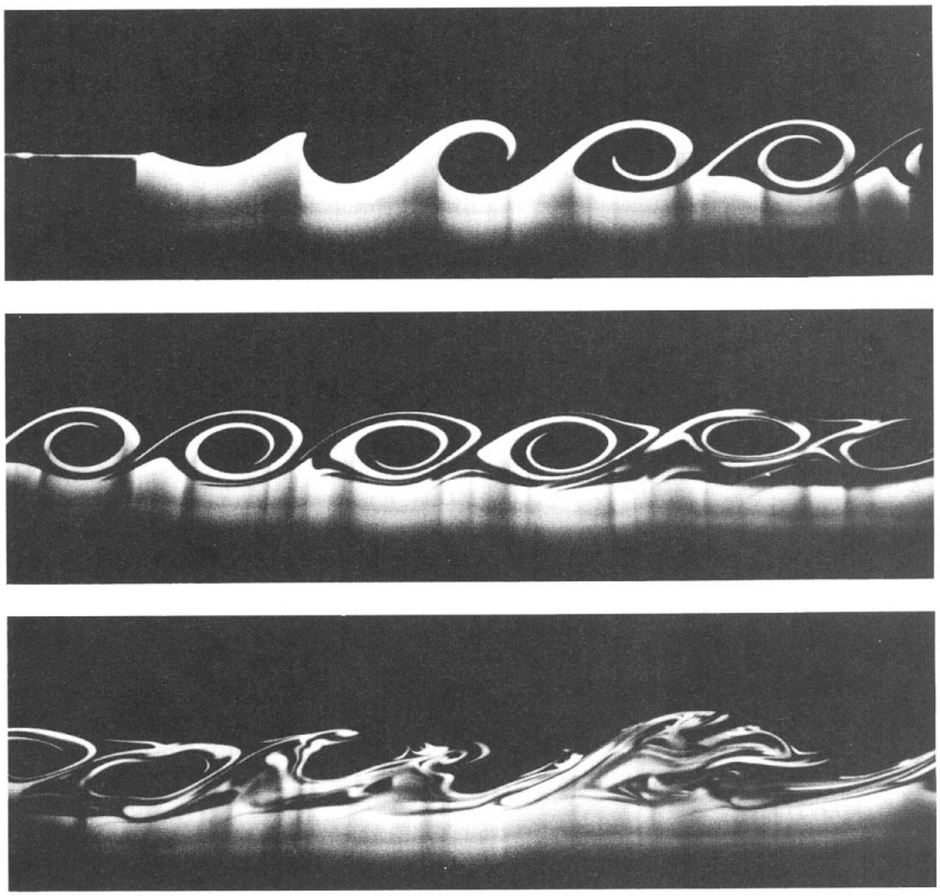

Figure 5.3: Laboratory demonstration of a Kelvin-Helmholtz instability as a boundary-value problem. Two layers flowing from left to right join downstream of a thin plate (visible on the left of the top photograph). The upper and faster moving fluid is slightly less dense than the fluid below. With downstream distance (from left to right on each photograph and from top to bottom panel), waves first turn into billows and later degenerate into turbulence. [Photo courtesy of Gregory A. Lawrence. For more details on the laboratory experiment, see Lawrence et al., 1991]

Figure 5.4: Kelvin-Helmholtz billows in the atmosphere made manifest by cloud formation. [Photo courtesy of Jean-Marie Beckers]

5.1.2 Finite-amplitude behavior

As waves grow, the small-amplitude theory fails, and the physics become more complicated. Based on general considerations such as volume conservation and a streamfunction expression for the flow inside billows depicted as partially flattened vortices, Scorer (1997, page 239) concluded that overturning billows have a length-to-height ration of about 2.7, from which we obtain that the height h of overturning is about 37% of the wavelength λ.

From the perspective of energy conservation, it is clear that the billowing motion involves lifting of denser fluid and sinking of lighter fluid, both of which cause a rise of the center of gravity which in turn demands a supply of potential energy. This potential energy can only be provided by conversion of kinetic energy present in the shear flow. Thus, the thickness h of overturning ultimately depends on the original velocity difference $\Delta\bar{u}$ and must be inversely proportional to both the gravitational acceleration g and the original density difference $\Delta\rho/\rho_0$. Thus,

$$h \;=\; C\,\frac{\rho_0\Delta\bar{u}^2}{g\Delta\rho}\;, \tag{5.29}$$

with the dimensionless constant of proportionality C indicating the efficiency of the process. Indeed, not all of the kinetic energy that is released goes into potential energy because some is consumed by turbulent dissipation. We shall return to the energy argument in the next chapter during our study of mixing.

Numerical simulations of the Kelvin-Helmholtz instability and sightings of cloud billows in the atmosphere abound (for example, Figure 5.4), and a cursory search

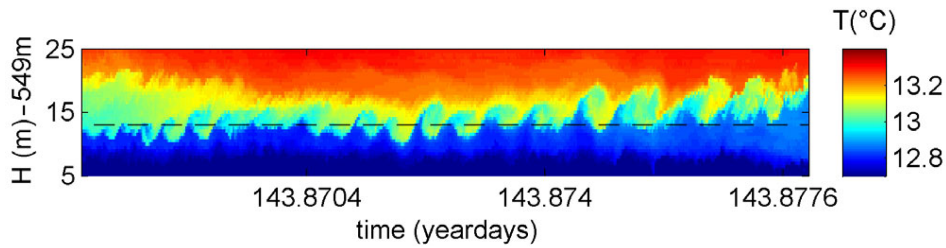

Figure 5.5: Kelvin-Helmholtz billows in the ocean as manifested by billows in the vertical distribution along a downslope tidal current. [From van Haren and Gostiaux, 2010]

of the web will quickly provide a good number of fine examples. Laboratory experiments of note were performed by Thorpe (1971) and De Silva et al. (1996), among many others.

Oceanic observations of Kelvin-Helmholtz billows are far less frequent due to the obvious difficulties associated with sub-surface observations. Woods (1968) is credited with the first report; Marmorino (1987) and Seim and Gregg (1994) were able to confirm the applicability of some of the theoretical criteria enunciated by Hazel (1972). A sighting of 50-meter tall Kelvin-Helmholtz billows in the deep ocean was also reported by van Haren and Gostiaux (2010) (Figure 5.5).

5.2 Instability of a Stratified Shear Flow

For the sake of mathematical expediency, the preceding section oversimplified the situation by considering a sharp discontinuity in density and velocity. In actual environmental systems, there exists a zone of gradual changes from one layer to the other, and the analysis becomes complicated because of the existence of non-constant coefficients. However, it is still possible to derive a useful result, namely a necessary but not sufficient condition for instability.

The situation now under consideration is that of a continuous stratification $\bar{\rho}(z)$, to which corresponds the stratification frequency $N(z)$ defined from

$$N^2 \;=\; -\,\frac{g}{\rho_0}\,\frac{d\bar{\rho}}{dz} \;>\; 0, \tag{5.30}$$

and accompanied by a sheared horizontal and unidirectional flow $\bar{u}(z)$. The equations governing a small perturbation of this system are:

$$\frac{\partial u'}{\partial x} \;+\; \frac{\partial w'}{\partial z} \;=\; 0 \tag{5.31}$$

$$\frac{\partial u'}{\partial t} \ + \ \bar{u}(z)\ \frac{\partial u'}{\partial x} \ + \ w'\ \frac{d\bar{u}}{dz} \ = \ - \ \frac{1}{\rho_0}\ \frac{\partial p'}{\partial x} \tag{5.32}$$

$$\frac{\partial w'}{\partial t} \ + \ \bar{u}(z)\ \frac{\partial w'}{\partial x} \ = \ - \ \frac{1}{\rho_0}\ \frac{\partial p'}{\partial z} \ - \ \frac{\rho' g}{\rho_0} \tag{5.33}$$

$$\frac{\partial \rho'}{\partial t} \ + \ \bar{u}(z)\ \frac{\partial \rho'}{\partial x} \ + \ w'\ \frac{d\bar{\rho}}{dz} \ = \ 0. \tag{5.34}$$

We introduce the streamfunction ψ by

$$u' \ = \ - \ \frac{\partial \psi}{\partial z}\ , \qquad w' \ = \ + \ \frac{\partial \psi}{\partial x}\ , \tag{5.35}$$

so that the first equation is automatically satisfied and then eliminate the pressure by cross-differentiation of the two momentum equations, (5.32)–(5.33), to obtain

$$\frac{\partial}{\partial t}\ \nabla^2 \psi \ + \ \bar{u}\ \frac{\partial}{\partial x}\ \nabla^2 \psi \ - \ \frac{d^2 \bar{u}}{dz^2}\ \frac{\partial \psi}{\partial x} \ = \ - \ \frac{g}{\rho_0}\ \frac{\partial \rho'}{\partial x}\ . \tag{5.36}$$

In terms of the streamfunction, the density equation (5.34) is:

$$\frac{\partial \rho'}{\partial t} \ + \ \bar{u}(z)\ \frac{\partial \rho'}{\partial x} \ + \ \frac{d\bar{\rho}}{dz}\ \frac{\partial \psi}{\partial x} \ = \ 0. \tag{5.37}$$

The velocity $\bar{u}$ and density $\bar{\rho}$ both vary in the vertical with z, but the coefficients of the preceding equations are invariant with respect to time and the horizontal. Thus, we can seek a solution in the wave form

$$\psi \ = \ \Psi(z)\ \mathrm{e}^{i(kx-\omega t)} \tag{5.38}$$

$$\rho' \ = \ R(z)\ \mathrm{e}^{i(kx-\omega t)}\ , \tag{5.39}$$

with real, positive wavenumber k and possibly complex frequency ω (see previous section). The last pair of equations reduces to

$$(\bar{u} - c) \left(\frac{d^2 \Psi}{dz^2} \ - \ k^2 \Psi \right) \ - \ \frac{d^2 \bar{u}}{dz^2}\ \Psi \ = \ - \ \frac{g}{\rho_0}\ R \tag{5.40}$$

$$(\bar{u} - c)\ R \ + \ \frac{d\bar{\rho}}{dz}\ \Psi \ = \ 0, \tag{5.41}$$

where the horizontal wave speed $c = \omega/k$ has been introduced for convenience. Finally, elimination of R between these two equations yields a single equation for $\Psi(z)$:

$$(\bar{u} - c) \left(\frac{d^2 \Psi}{dz^2} \ - \ k^2 \Psi \right) \ + \ \left(\frac{N^2}{\bar{u} - c} \ - \ \frac{d^2 \bar{u}}{dz^2} \right)\ \Psi \ = \ 0. \tag{5.42}$$

This is the *Taylor-Goldstein equation*, which governs the vertical structure of the wave perturbation. In writing it, we expressed the vertical density gradient in terms of N^2 according to (5.30). For rigid boundaries on top and bottom of the system, impermeability ($w = 0$) requires that Ψ vanish there.

5.2.1 Richardson number criterion

With its z–dependent coefficients, this equation is very difficult to solve even after the velocity and density profiles are specified (linear, hyperbolic tangent or other). There are, however, a couple of results that can be obtained in the general case of unspecified velocity and density profiles. Toward this end, it is first noted that, while the wave speed c and the streamfunction Ψ may be complex quantities, the equation does not include explicitely the imaginary unit i. This implies that, if $\Psi(z)$ is a solution associated with the wave speed c, the complex conjugate function $\Psi^*(z)$ is another solution associated with the complex conjugate wave speed c^*. To see this, it suffices to take the complex conjugate of Equation (5.42). Complex c values come in conjugate pairs and there is always one of the two with a positive imaginary value, leading to a positive value of ω_i. In other words, a non-zero c_i is symptomatic of instability.

Next, we substitute the new variable:

$$\phi \;=\; \frac{\Psi}{\sqrt{\bar{u}-c}}\;, \tag{5.43}$$

which transforms the equation into

$$\frac{d}{dz}\left[(\bar{u}-c)\,\frac{d\phi}{dz}\right] \;-\; \left[k^2(\bar{u}-c) \;+\; \frac{1}{2}\,\frac{d^2\bar{u}}{dz^2} \;+\; \frac{(d\bar{u}/dz)^2-4N^2}{4(\bar{u}-c)}\right]\,\phi \;=\; 0. \tag{5.44}$$

After multiplication by the complex conjugate ϕ^* and vertical integration across the domain, the equation can be manipulated to yield:

$$\int(\bar{u}-c)\left(\left|\frac{d\phi}{dz}\right|^2+k^2|\phi|^2\right)\,dz \;+\; \frac{1}{2}\int\frac{d^2\bar{u}}{dz^2}\,|\phi|^2\,dz$$
$$+\;\frac{1}{4}\int\frac{(d\bar{u}/dz)^2-4N^2}{(\bar{u}-c)}\,|\phi|^2\,dz \;=\; 0. \tag{5.45}$$

The second integral is always real, and the imaginary part of the equation therefore reduces to the imaginary parts of the first and last integrals, which together imply:

$$c_i\int\left(\left|\frac{d\phi}{dz}\right|^2+k^2|\phi|^2\right)\,dz \;=\; \frac{c_i}{4}\int\frac{(d\bar{u}/dz)^2-4N^2}{|\bar{u}-c|^2}\,|\phi|^2\,dz. \tag{5.46}$$

The discussion then proceeds as follows. Either c_i is zero or it is not. If it is zero, ω is real and the wave perturbation is stable. So, instability can only occur if c_i is non-zero, which requires that the two integrals in (5.46) be equal to each other and since the first is always positive, the second must be, too. This can only occur if $(d\bar{u}/dz)^2 - 4N^2$ is positive in at least some portion of the domain. In other words, the inequality

$$\left(\frac{d\bar{u}}{dz}\right)^2 \;>\; 4N^2 \tag{5.47}$$

is a required condition before instability can occur. Note that this is a necessary but not sufficient condition for instability. Put another way, the flow is stable if $(d\bar{u}/dz)^2$ is smaller than $4N^2$ throughout the domain, and the flow may or may not be unstable if the maximum of $(d\bar{u}/dz)^2$ exceeds $4N^2$ anywhere in the domain.

If the velocity $\bar{u}(z)$ and density $\bar{\rho}(z)$ are linear in z, i.e. if the shear and stratification are uniform, $d\bar{u}/dz = \Delta U/H$ and $N^2 = g\Delta\rho/\rho_0 H$, where ΔU and $\Delta\rho$ are respectively the velocity and density differences across the vertical extent H of the domain. The condition necessary for instability can then be stated in terms of the Richardson number:

$$Ri \;=\; \frac{gH\Delta\rho}{\rho_0 \Delta U^2} \;<\; \frac{1}{4}\;. \tag{5.48}$$

To generalize the conclusion to arbitrary velocity and density profiles, we define the *gradient Richardson number*:

$$Ri_g(z) \;=\; \frac{N^2}{(d\bar{u}/dz)^2}\;, \tag{5.49}$$

and the system is stable if $Ri_g(z)$ lies above 0.25 for all values of z.

The condition that the minimum of Ri_g with respect to z fall below 0.25 is necessary but not sufficient for instability. But, while there are theoretical examples of systems for which the actual instability threshold is a value less than 0.25, numerical studies (Hazel, 1972) indicate that the criterion

$$Ri_g \;<\; \frac{1}{4} \tag{5.50}$$

is a useful method of identification of unstable stratified shear flows.

Occasionally, the density stratification and velocity shear do not occur on the same vertical scale, and different cases are possible. Holmboe (1962) investigated the stability of two-layer flows with a thin density interface relative to the velocity shear layer thickness. Such flows occur in estuaries because of the relatively low molecular diffusivity of salt compared to viscosity. He found that at low Richardson numbers the instability is similar to the Kelvin-Helmholtz instability, but that at higher Richardson numbers arises a second mode of instability (now known as *Holmboe's instability*) consisting of two trains of interfacial waves traveling at the same speed but in opposite directions with respect to the mean flow. Laboratory experiments of Holmboe's instability were performed by Zhu and Lawrence (2001).

5.2.2 Bounds on wave speeds and growth rates

The preceding analysis taught us that instabilities may occur when certain conditions are met. A question then naturally arises: If the flow is unstable, how fast will perturbations grow? In the general case of an arbitrary shear flow $\bar{u}(y)$, a precise determination of the growth rate of unstable perturbations is not possible. Nonetheless, some practical result that can be derived from the Taylor-Goldstein equation without solving it exactly. For this, we now substitute the variable[3]

$$a \ = \ \frac{\Psi}{\bar{u} - c} \ , \tag{5.51}$$

which turns Equation (5.42) into

$$\frac{d}{dz} \left[(\bar{u} - c)^2 \ \frac{da}{dz} \right] \ - \ k^2 (\bar{u} - c)^2 a \ + \ N^2 a \ = \ 0. \tag{5.52}$$

Multiplication by the complex conjugate a^* and integration across the domain then yield

$$\int (\bar{u} - c)^2 \left(\left| \frac{da}{dz} \right|^2 + k^2 |a|^2 \right) \ dz \ = \ \int N^2 \ |a|^2 \ dz. \tag{5.53}$$

The imaginary part of this expression

$$2c_i \int (\bar{u} - c_r) \left(\left| \frac{da}{dz} \right|^2 + k^2 |a|^2 \right) \ dz \ = \ 0 \tag{5.54}$$

implies that, if there is instability ($c_i \neq 0$), the phase speed c_r must lie between the extrema of $\bar{u}(z)$:

$$U_{\min} \ < \ c_r \ < \ U_{\max}. \tag{5.55}$$

Physically, the wavy perturbation, if unstable, must travel with a speed that matches that of the entraining flow, in at least one location. In other words, there will always be a place in the domain where the wave does not drift with respect to the ambient flow and grows in place. It is precisely this local coupling between wave and flow that allows the wave to extract energy from the flow and to grow at its expense. The location where the phase speed is equal to the flow velocity is called a *critical level*.

The real part of (5.53) holds information on the imaginary part c_i. This real part is

$$\int [(\bar{u} - c_r)^2 - c_i^2] \left(\left| \frac{da}{dz} \right|^2 + k^2 |a|^2 \right) \ dz \ = \ \int N^2 \ |a|^2 \ dz \ > \ 0. \tag{5.56}$$

Adding to this the following, obvious inequality:

$$\int (\bar{u} - U_{\min})(U_{\max} - \bar{u}) \left(\left| \frac{da}{dz} \right|^2 + k^2 |a|^2 \right) \ dz \ > \ 0, \tag{5.57}$$

[3]It can be shown that this variable is the vertical displacement of fluid particles caused by the wave perturbation.

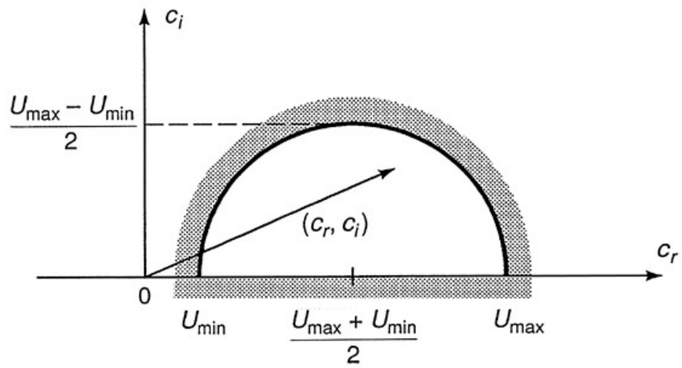

Figure 5.6: The semicircle theorem. Growing perturbations of wavenumber k must have phase speeds c_r and growth rates kc_i such that the tip of the vector $(c_r,\, c_i)$ falls within the half-circle constructed from the minimum and maximum velocities of the ambient shear flow $\bar{u}$, as depicted in the figure.

and then subtracting from the result Equation (5.54) premultiplied by $(U_{\min} + U_{\max} - 2c_r)$, we obtain

$$\left[\left(c_r - \frac{U_{\min} + U_{\max}}{2} \right)^2 + c_i^2 - \left(\frac{U_{\max} - U_{\min}}{2} \right)^2 \right] \int \left(\left| \frac{da}{dz} \right|^2 + k^2 |a|^2 \right) \, dz \; \leq \; 0.$$

Because the integral can only be positive, the preceding bracketed quantity must be negative:

$$\left(c_r - \frac{U_{\min} + U_{\max}}{2} \right)^2 \; + \; c_i^2 \; \leq \; \left(\frac{U_{\max} - U_{\min}}{2} \right)^2 . \tag{5.58}$$

This inequality implies that, in the complex plane, the number $c_r + ic_i$ must lie within the circle centered at $[(U_{\min} + U_{\max})/2, 0]$ and of radius $(U_{\max} - U_{\min})/2$. Since we are interested in modes that grow in time, c_i is positive, and only the upper half of that circle is relevant (Figure 5.6). This result is called Howard's semicircle theorem[4].

It is readily evident from inequality (5.58) or Figure 5.6 that c_i is bounded above by

$$c_i \;\leq\; \frac{U_{\max} - U_{\min}}{2} \; . \tag{5.59}$$

The perturbation's growth rate kc_i is thus likewise bounded above by $k(U_{\max} - U_{\min})/2$.

[4]In honor of Louis N. Howard (1929–), American mathematician and fluid dynamicist

5.3 Barotropic Instability

Sections 5.1 and 5.2 demonstrated that a velocity shear can destabilize an otherwise stable stratification. It follows that a velocity shear can also destabilize a non-stratified system. In particular, a horizontal velocity shear in a two-dimensional horizontal flow should be destabilizing. The stability theory of homogeneous shear flows is a well developed chapter in fluid mechanics (see Kundu, 1990, Section 11.9). Here, we generalize the problem by introducing the Coriolis force.

Our present assumptions are that the fluid is homogeneous and inviscid, and that its flow is confined to the two-dimensional horizontal plane. The Coriolis parameter is retained but not allowed to vary (under the assumption of a narrow latitudinal range). The governing equations are (Section 3.3):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.60}$$

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho_0}\,\frac{\partial p}{\partial x} \tag{5.61}$$

$$\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho_0}\,\frac{\partial p}{\partial y}\,. \tag{5.62}$$

For the basic state, we choose a zonal current with arbitrary meridional profile: $u = \bar{u}(y)$, $v = 0$. This is an exact solution to the nonlinear equations as long as the pressure force balances the Coriolis force[5]:

$$f\bar{u}(y) = -\frac{1}{\rho_0}\,\frac{d\bar{p}}{dy}\,. \tag{5.63}$$

Next, we superimpose a small perturbation, meant to represent an arbitrary wave of weak amplitude:

$$u = \bar{u}(y) + u'(x,y,t) \tag{5.64}$$

$$v = v'(x,y,t) \tag{5.65}$$

$$p = \bar{p}(y) + p'(x,y,t), \tag{5.66}$$

where the perturbations u', v' and p' are taken to be much smaller that the corresponding variables of the basic flow (i.e., u' and v' much less than $\bar{u}$, and p' much less than $\bar{p}$). Substitution in Equations (5.60), (5.61) and (5.61), and subsequent linearization yield:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \tag{5.67}$$

$$\frac{\partial u'}{\partial t} + \bar{u}\,\frac{\partial u'}{\partial x} + v'\,\frac{d\bar{u}}{dy} - fv' = -\frac{1}{\rho_0}\,\frac{\partial p'}{\partial x} \tag{5.68}$$

[5] in the absence of a Coriolis force, the pressure field is uniform, while the $\bar{u}$ velocity component may be an arbitrary function of y.

$$\frac{\partial v'}{\partial t} + \bar{u}\,\frac{\partial v'}{\partial x} + f u' = -\,\frac{1}{\rho_0}\,\frac{\partial p'}{\partial y}\,. \tag{5.69}$$

Equation (5.67) admits the streamfunction ψ, defined as

$$u' = -\,\frac{\partial \psi}{\partial y}\,, \quad v' = +\,\frac{\partial \psi}{\partial x}\,. \tag{5.70}$$

The choice of signs corresponds to a flow along streamlines with the higher streamfunction values on the right.

A cross-differentiation of the momentum equations (5.68) and (5.69) and the elimination of the velocity components in terms of the streamfunction leads to a single equation for this streamfunction:

$$\left(\frac{\partial}{\partial t} + \bar{u}\,\frac{\partial}{\partial x}\right)\,\nabla^2\,\psi - \frac{d^2\bar{u}}{dy^2}\,\frac{\partial \psi}{\partial x} = 0. \tag{5.71}$$

This equation has coefficients that depend on $\bar{u}$ and, therefore, on the transverse coordinate y only. A wave form in the direction of the flow is a solution:

$$\psi(x,y,t) = \Psi(y)\,\mathrm{e}^{i(kx-\omega t)}\,. \tag{5.72}$$

Substitution provides the following second-order ordinary differential equation for the amplitude $\Psi(y)$:

$$\frac{d^2\Psi}{dy^2} - k^2\Psi - \frac{d^2\bar{u}/dy^2}{\bar{u}(y) - c}\,\Psi = 0, \tag{5.73}$$

where $c = \omega/k$ is the speed of wave propagation. An equation of this type is called the *Rayleigh equation*[6] and it is similar to the Taylor-Goldstein equation (5.42) encountered in Section 5.2. The key features are the non-constant coefficient in the third term and the fact that its denominator may be zero, creating a singularity.

Let us now consider the attending boundary conditions. For simplicity, consider channel flow contained between two striaght walls, at $y = 0$ and L. The atmospheric jet stream in the upper troposphere, the Gulf Stream after its seaward turn off Cape Hatteras (36°N), and the Antarctic Circumpolar Current are all good examples of such flow, except that the lateral boundaries are not rigid but rather soft.

If the boundaries prevent fluid from entering and leaving the channel, v' is zero there, and (5.70) implies that the streamfunction must be a constant along each wall. In other words, walls are streamlines. This is possible only if the wave amplitude obeys

$$\Psi(y=0) = \Psi(y=L) = 0. \tag{5.74}$$

The second-order, homogeneous problem of (5.73) and (5.74) can be viewed as an eigenvalue problem: The solution is trivial ($\Psi = 0$), unless the phase velocity

[6]in honor of John W. Strutt, Lord Rayleigh (1842–1919), famous English mathematician and experimental physicist

assumes a specific value (eigenvalue), in which case a nonzero function Ψ can be determined within an arbitrary multiplicative constant.

In general, the eigenvalues c may be complex. If c admits the function Ψ, then the complex conjugate c^* admits the complex conjugate function Ψ^* and is thus another eigenvalue. This can be readily verified by taking the complex conjugate of equation (5.73). Hence, complex eigenvalues come in pairs.

Because mathematical difficulties prevent a general determination of the c values for an arbitrary velocity profile $\bar{u}(y)$ (the analysis is difficult even for idealized but nontrivial profiles), we shall not attempt to solve the problem (5.73)–(5.74) exactly but will instead establish some of its integral properties and, in so doing, reach weaker stability criteria.

If we multiply equation (5.73) by Ψ^* and then integrate across the domain, we obtain

$$- \int \left(\left| \frac{d\Psi}{dy} \right|^2 \; + \; k^2 |\Psi|^2 \right) \; dy \; - \; \int \frac{d^2\bar{u}/dy^2}{\bar{u} - c} \; |\Psi|^2 \; dy \; = \; 0, \tag{5.75}$$

after an integration by parts. The imaginary part of this expression is

$$c_i \; \int \frac{d^2\bar{u}}{dy^2} \; \frac{|\Psi|^2}{|\bar{u} - c|^2} \; dy \; = \; 0. \tag{5.76}$$

Two cases are possible: Either c_i vanishes or the integral does. If c_i is zero, the basic flow admits no growing disturbance and is stable. But, if c_i is not zero, then the integral must vanish, which requires that the second derivative $d^2\bar{u}/dy^2$ of the velocity must change sign at least once within the domain. Summing up, we conclude that a necessary condition for instability is that expression the second derivaztive of the velocity $\bar{u}$ vanish somewhere inside the domain. Conversely, a sufficient condition for stability is that the second derivaztive of the velocity $\bar{u}$ not vanish anywhere within the domain (on the boundaries maybe, but not inside the domain). Physically, the vorticity of the basic flow, $f - d\bar{u}/dy$, must reach an extremum within the domain to cause instabilities. This result was first derived by Kuo (1949).

This first criterion can be strengthened by considering next the real part of (5.75):

$$- \int (\bar{u} - c_r) \; \frac{d^2\bar{u}}{dy^2} \; \frac{|\Psi|^2}{|\bar{u} - c|^2} \; dy \; = \; \int \left(\left| \frac{d\Psi}{dy} \right|^2 \; + \; k^2 |\Psi|^2 \right) \; dy. \tag{5.77}$$

In the event of instability, the integral in (5.76) vanishes. Multiplying it by $(c_r - \bar{u}_0)$, where $\bar{u}_0$ is any real constant, adding the result to (5.77), and noting that the right-hand side of (5.77) is always positive for non-zero perturbations, we obtain:

$$- \int (\bar{u} - \bar{u}_0) \; \frac{d^2\bar{u}}{dy^2} \; \frac{|\Psi|^2}{|\bar{u} - c|^2} \; dy \; > \; 0. \tag{5.78}$$

This inequality demands that the expression

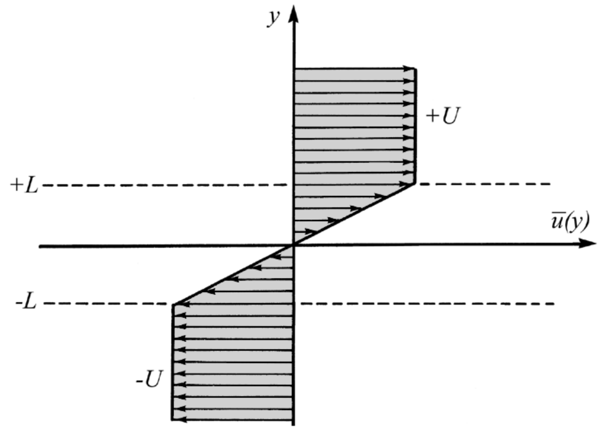

Figure 5.7: An academic shear-flow profile that lends itself to analytic treatment. This profile meets both necessary conditions for instability and is found to be unstable to long waves.

$$(\bar{u} - \bar{u}_0)\, \frac{d^2\bar{u}}{dy^2} \tag{5.79}$$

be negative in at least some finite portion of the domain. Because this must hold true for any constant $\bar{u}_0$, it must be true in particular if $\bar{u}_0$ is the value of $\bar{u}(y)$ where $d^2\bar{u}/dy^2$ vanishes. Hence, a stronger criterion is: Necessary conditions for instability are that $d^2\bar{u}/dy^2$ vanish at least once within the domain *and* that $(\bar{u}-\bar{u}_0)(d^2\bar{u}/dy^2)$, where $\bar{u}_0$ is the value of $\bar{u}(y)$ at which the first expression vanishes, be negative in at least some finite portion of the domain. Although this stronger criterion offers still no sufficient condition for instability, it is generally quite useful.

Howard's semicircle theorem holds again.

5.3.1 A simple example

The preceding considerations on the existence of instabilities and their properties are rather abstract. So, let us work out an example to illustrate the concepts. For simplicity, we take a shear flow that is piecewise linear (Figure 5.7):

$$y \;<\; -L: \qquad \bar{u} \;=\; -U, \quad \frac{d\bar{u}}{dy} \;=\; 0, \quad \frac{d^2\bar{u}}{dy^2} \;=\; 0 \tag{5.80}$$

$$-L \;<\; y \;<\; +L: \qquad \bar{u} \;=\; \frac{U}{L}\, y, \quad \frac{d\bar{u}}{dy} \;=\; \frac{U}{L}\,, \quad \frac{d^2\bar{u}}{dy^2} \;=\; 0 \tag{5.81}$$

$$+L \;<\; y: \qquad \bar{u} \;=\; +U, \quad \frac{d\bar{u}}{dy} \;=\; 0, \quad \frac{d^2\bar{u}}{dy^2} \;=\; 0, \tag{5.82}$$

where U is a positive constant and the domain width is now infinity. Although the second derivative vanishes within each of the three segments of the domain, it is nonzero at their junctions. As y increases, the first derivative $d\bar{u}/dy$ changes from zero to a positive value and back to zero, so it can be said that the second derivative is positive at the first junction ($y = -L$) and negative at the second ($y = +L$). It thus changes sign in the domain, and this satisfies the first condition for the existence of instabilities, namely, that $d^2\bar{u}/dy^2$ vanish within the domain. The second condition, that expression (10.24), now reduced to

$$- \bar{u} \frac{d^2\bar{u}}{dy^2} ,$$

be positive in some portion of the domain, is also satisfied because $d^2\bar{u}/dy^2$ has the sign opposite to $\bar{u}$ at each junction of the profile. So, although instabilities are not guaranteed to exist, we may not rule them out.

We now proceed with the solution. In each of the three domain segments, governing equation (5.73) reduces to

$$\frac{d^2\Psi}{dy^2} - k^2 \Psi = 0, \tag{5.83}$$

and admits solutions of the type $\exp(+ky)$ and $\exp(-ky)$. This yields two constants of integration per domain segment, for a total of six. Six conditions are then applied. First, Ψ is required to vanish at large distances:

$$\Psi(-\infty) = \Psi(+\infty) = 0.$$

Next, continuity of the meridional displacements at $y = \pm L$ requires, by virtue of (10.26) and by virtue of the continuity of the $\bar{u}(y)$ profile, that Ψ, too, be continuous there:

$$\Psi(-L - \epsilon) = \Psi(-L + \epsilon), \quad \Psi(+L - \epsilon) = \Psi(+L + \epsilon),$$

for arbitrarily small values of ϵ. Finally, the integration of governing equation (10.16) across the lines joining the domain segments

$$\int_{\pm L-\epsilon}^{\pm L+\epsilon} \left[(\bar{u} - c) \frac{d^2\Psi}{dy^2} - k^2 (\bar{u} - c) \Psi - \frac{d^2\bar{u}}{dy^2} \Psi \right] dy = 0,$$

followed by an integration by parts, implies that

$$(\bar{u} - c) \frac{d\Psi}{dy} - \frac{d\bar{u}}{dy} \Psi$$

must be continuous at both $y = -L$ and $y = +L$. Applying these six conditions leads to a homogeneous system of equations for the six constants of integration. Nonzero perturbations exist when this system admits a nontrivial solution – that is, when its determinant vanishes. Some tedious algebra yields

$$\frac{c^2}{U^2} = \frac{(1-2kL)^2 - e^{-4kL}}{(2kL)^2} . \tag{5.84}$$

Equation (10.40) is the dispersion relation, providing the wave velocity c in terms of the wave number k and the flow parameters L and U. It yields a unique and real c^2, either positive or negative. If it is positive, c is real and the perturbation behaves as a non-amplifying wave. But, if c^2 is negative, c is imaginary and one of the two solutions yields an exponentially growing mode [a proportional to $\exp(kc_it)$]. Obviously, the instability threshold is $c^2 = 0$, in which case the dispersion relation (10.40) yields $kL = 0.639$. There is thus a critical wave number $k = 0.639/L$ or critical wavelength $2\pi/k = 9.829L$ separating stable from unstable waves. It can be shown by inspection of the dispersion relation that shorter waves ($kL > 0.639$) travel without growth (because $c_i = 0$), whereas longer waves ($kL < 0.639$) grow exponentially without propagation (because $c_r = 0$). In sum, the basic shear flow is unstable to long-wave disturbances.

An interesting quest is the search for the fastest growing wave, because this is the dominant wave, at least until finite-amplitude effects become important and the preceding theory loses its validity. For this, we look for the value of kL that maximizes kc_i, where c_i is the positive imaginary root of (10.40). The answer is $kL = 0.398$, from which follows the wavelength of the fastest growing mode:

$$\lambda_{\text{fastest growth}} = \frac{2\pi}{k} = 15.77\ L = 7.89\ (2L). \tag{5.85}$$

This means that the wavelength of the perturbation that dominates the early stage of instability is about 8 times the width of the shear zone. Its growth rate is

$$(kc_i)_{\max} = 0.201\ \frac{U}{L} , \tag{5.86}$$

corresponding to $c_i = 0.505U$. It is left to the reader as an exercise to verify the preceding numerical values.

At this point, it is instructive to unravel the physical mechanism responsible for the growth of long-wave disturbances. Figure 10-3 displays the basic flow field, on which is superimposed a wavy disturbance. The phase shift between the two lines of discontinuity is that propitious to wave amplification. As the middle fluid, endowed with a clockwise vorticity, intrudes in either neighboring strip, where the vorticity is nonexistent, it produces local vorticity anomalies, which can be viewed as vortices. These vortices generate clockwise rotating flows in their vicinity, and, if the wavelength is sufficiently long, the distance between the two lines of discontinuity appears relatively small and the vortices from each side interact with those on the other side. Under a proper phase difference, such as the one depicted on Figure 5.8, the vortices entrain one another further into the regions of no vorticity, thereby amplifying the crests and troughs of the wave. The wave amplifies, and the basic shear flow cannot persist. As the wave grows, nonlinear terms are no longer negligible, and some level of saturation is reached. The ultimate state (Figure 5.8) is that of a series of clockwise vortices embedded in a weakened ambient shear flow (Zabusky et al., 1979; Dritschel, 1989).

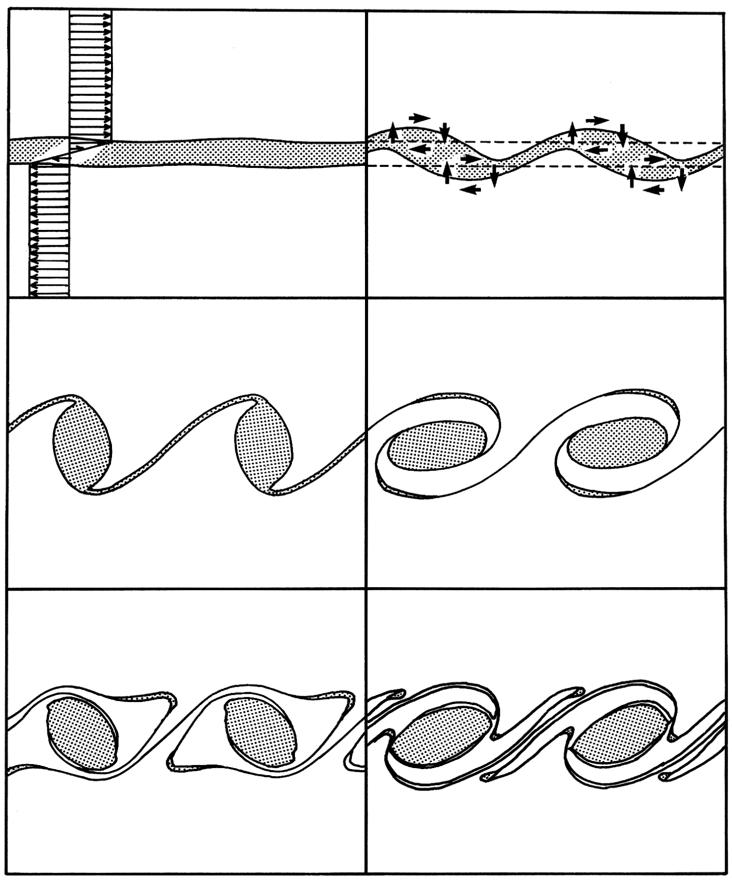

Figure 5.8: Finite-amplitude development of the instability of the shear flow depicted in Figure 5.7 as computed by the method of contour dynamics. The throughs and crests of the wave induce a vortex field, which, in turn, amplifies those throughs and crests. The wave does not travel but amplifies with time.

5.4 Inertial and Baroclinic Instability

Two additional forms of instabilities exist when the effects of the earth's rotation are included in the dynamics. Both consider a unidirectional but oblique shear flow $v(x,z)$ in balance with an oblique density stratification $\rho(x,z)$ via relation the geostrophic relation

$$f\,\frac{\partial v}{\partial z} \;=\; -\,\frac{g}{\rho_0}\,\frac{\partial \rho}{\partial x}\;, \tag{5.87}$$

in which f is the Coriolis parameter, defined in (3.29) and proportional to the rotation rate of the earth, g is the gravitational acceleration, and ρ_0 is the reference density under the Boussinesq approximation.

The first type of instability, called *inertial instability*, is akin to gravitational instability but with the Coriolis force combining with gravity. Its analysis (Cushman-Roisin and Beckers, 2011, Section 17.2) concludes that the preceding type of flow is stable only if all three of the following inequalities are satisfied:

$$F^2 \;\geq\; 0, \quad N^2 \;\geq\; 0, \quad \text{and} \quad F^2N^2 \;\geq\; f^2M^2, \tag{5.88}$$

in which

$$F^2 = f\,\left(f + \frac{\partial v}{\partial x}\right), \quad N^2 = -\frac{g}{\rho_0}\,\frac{\partial \rho}{\partial z}, \quad \text{and} \quad M = \frac{\partial v}{\partial z} = -\frac{g}{f\rho_0}\,\frac{\partial \rho}{\partial x}\;. \tag{5.89}$$

In the second inequality one recognizes the condition for gravitational stability (light fluid on top of denser fluid), whereas the first and third inequalities are peculiar to rotational effects. Should any of the these three conditions not be met, the flow undergoes immediate chaotic mixing.

Should all three conditions (5.88) be met, the flow is said to be inertially stable but may still be vulnerable to a slower and more organized form of instability, called *baroclinic instability.* The latter instability has been identified as the cause of mid-latitude weather systems, and its mechanism can be explained as the self amplification of a wave perturbation with suitable phase shift in the vertical (Cushman-Roisin and Beckers, 2011, Section 17.3). The analysis is complicated, and results are found to be dependent on both wavelength of perturbation and the type of boundary conditions imposed on the flow. In particular, a sloping boundary can either stabilize or destabilize the flow depending on its tilt with respect to the oblique density stratification.

Problems

5-1. Prove the assertion made below Equation (5.27) that the growth rate of unstable K-H waves increases with decreasing wavelength below the critical value.

5-2. At which speed do unstable (growing) Kelvin-Helmholtz waves travel? Are they dispersive?

5-3. The nighttime atmosphere above a plain consists in a lower layer with temperature of 9°C and weak wind of 2 m/s, overlaid by an upper layer of air with temperature of 16°C and wind of 12 m/s blowing in the same direction. A sharp interface between the two layers should be unstable, and the system therefore adopts a smooth transition from one layer to the other. Assuming that this layer of transition is characterized by a linear variation in both temperature and velocity and by a critical Richardson number, determine its thickness.

5-4. In a 20m deep, thermally stratified lake, the stratification frequency N is equal to 0.02 s^{-1}, and there exist internal waves with vertical wavelength equal to twice the depth (first seiche mode). By their very nature, these waves possess a certain amount of horizontal velocity shear (du/dz not zero) and, if this shear becomes excessive (according to the gradient Richardson number criterion), overturning and mixing takes place. What is the amplitude U of the horizontal velocity in the waves that triggers overturning and mixing?

5-5. In Loch Ness, Scotland, suppose that what appears to be the underwater serpent is actually a manifestation of a Kelvin-Helmholtz instability at the interface between a clear surface water layer flowing at 5 cm/s and a dark, denser water layer at rest below. If the observed wavelength is about 3.5 m, estimate the relative density difference $\Delta\rho/\rho_0$ between the two layers.

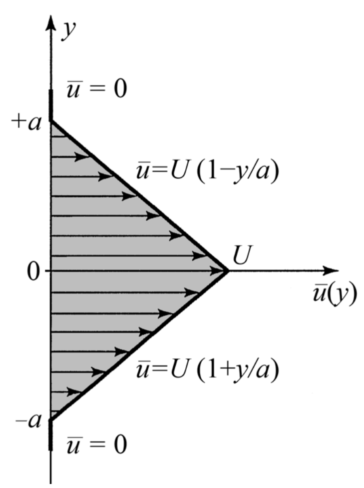

Figure 5.9: A jet-like profile (for Problems 5-8 and 5-9).

5-6. The water density distribution near a vertical boundary is oblique and given by:

$$\rho(x, z) = \rho_0 \, (1 + ax - bz), \tag{5.90}$$

with $b = 2.8 \times 10^{-4}$ m^{-1}. It is in geostrophic equilibrium with a vertically sheared current $v(z)$ according to (5.87) with Coriolis parameter $f = 9.94 \times 10^{-5}$ s^{-1}. This current vanishes at depth $z = -20$m. What is the value of the current at the surface ($z = 0$)? What is the critical value of a with respect to inertial instability? What is the Richardson number then?

5-7. Generalize the finding of the previous problem and show that any vertically sheared flow $v(z)$ in geostrophic balance with an oblique density distribution $\rho(x, z)$ is inertially stable as long as its Richardson number is higher than a certain threshold value. What is this threshold value? Is such a flow more easily vulnerable to inertial instability (Section 5.4) or shear instability (Section 5.2)?

5-8. Derive the dispersion relation and establish a stability threshold for the jet-like velocity profile of Figure 5.9.

5-9. Redo the preceding problem with the jet occupying a channel between $y = -a$ and $y = +a$.