Environmental Transport and Fate

Chapter 2

–

Diffusion – Part 5:

With advection

Benoit Cushman-Roisin
Thayer School of Engineering
Dartmouth College

Oftentimes, the fluid within which diffusion takes place is also moving in a preferential direction. The obvious cases are those of a flowing river and of a smokestack plume being blown by the wind.

Formulation of the problem

We now retain the advective flux and combine it with the diffusive flux.
Recall that in one dimension we established the total flux, at 1D, as

$$q = c\,u - D\frac{\partial c}{\partial x}$$

1D budget becomes

$$\frac{\partial c}{\partial t} = -\frac{\partial q}{\partial x} \quad \rightarrow \quad \frac{\partial c}{\partial t} = -\frac{\partial (cu)}{\partial x} + \frac{\partial}{\partial x}\left(D\frac{\partial c}{\partial x}\right)$$

At constant values of u and D, this may be rewritten as

$$\boxed{\underbrace{\frac{\partial c}{\partial t}}_{\text{accumulation}} + \underbrace{u\frac{\partial c}{\partial x}}_{\text{advection}} = \underbrace{D\frac{\partial^2 c}{\partial x^2}}_{\text{diffusion}}}$$

The solution for the prototypical case of an instantaneous (at $t = 0$) and localized (at $x = 0$) release is:

Note the shift from x to $x - ut$, as if the origin were moving in time at speed u .

$$c(x,t) = \frac{M}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x-ut)^2}{4Dt} \right)$$

If decay is also present, making the situation one of simultaneous advection-diffusion-decay, the budget equation is:

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = D\frac{\partial^2 c}{\partial x^2} - Kc$$

and the prototypical solution for an instantaneous and localized release is:

$$c(x,t) = \frac{M}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x-ut)^2}{4Dt} \underbrace{- Kt}_{\text{decay}} \right)$$

Higher dimensions

At 2D, with velocity vector (u, v) along axis directions x and y:

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \right)$$

At 3D, with velocity vector (u, v, w) along axis directions x, y and z:

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \right)$$

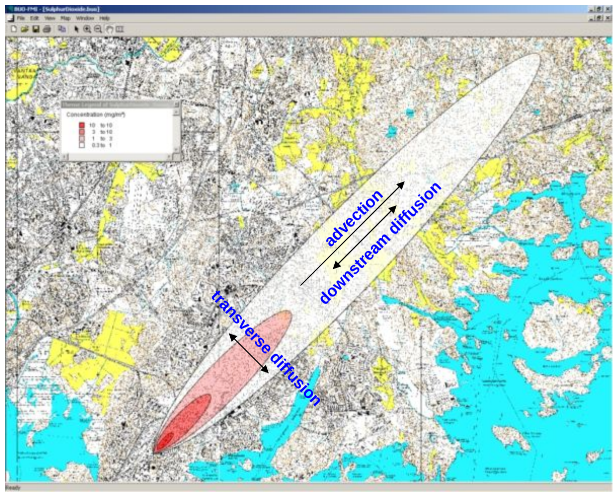
Note: An advection direction may not be active at the same time as diffusion in the same direction.

Example at 2D: If the x-direction is taking as the wind direction, there is no advection in the y-direction ($v = 0$), but there may still be diffusive spreading in that direction. The budget equation is then:

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \right)$$

Example of advection-diffusion in the atmosphere:

Simulation of the effect of a point-source in Finland



Simulation by the Finnish Meteorological Institute

Difference between advection and diffusion

Both advection and diffusion move the pollutant from one place to another,
but each accomplishes this differently.

The essential difference is:

- Advection goes one way (downstream);
- Diffusion goes both ways (regardless of a stream direction).

This is seen in the respective mathematical expressions:

- Advection $u\partial c/\partial x$ has a first-order derivative,
  which means that if x is replaced by $-x$ the term changes signs (anti-symmetry);
- Diffusion $D\partial^2 c/\partial x^2$ has a second-order derivative,
  which means that if x is replaced by $-x$ the term does not change sign (symmetry).

Relative importance of advection with respect to diffusion

The following question arises:

If both advection and diffusion are capable of displacing the pollutant, albeit in different ways, in which condition is one more effective than the other?

That is, can we have cases of fast advection and relatively weak diffusion and other cases of fast diffusion and negligible advection?

To answer this question, we must compare the sizes of the $u\partial c/\partial x$ and $D\partial c^2/\partial x^2$ terms to each other, and this is accomplishes by introducing "scales".

A **scale** is a quantity of dimension identical to the variable to which it refers and the value of which gives a practical estimate of the magnitude of that variable.

Examples:
scale for the width of the Mississippi River is L = 100 m,
scale for mid-ocean depths is H = 3000 m,
scale of the prevailing winds in the atmosphere is U = 10 m/s,
scale for the concentration of a substance in a finite domain could be taken as the average or maximum concentration value.

To make matters easy, scales are usually taken as pure constants (independent of space and time), and their values are rounded to just one or a few digits.

VARIABLE	SCALE	CHOICE OF VALUE
c	C	Typical concentration value, such as average, initial or boundary value
u	U	Typical velocity value, such as maximum value
x	L	Approximate domain length or size of release location

Using these scales, we can derive estimates of the sizes of the different terms.

Since the derivative $\partial c/\partial x$ is expressing, after all, a difference in concentration over a distance (in the infinitesimal limit), we can estimate it to be approximately (within 100% or so, but certainly not completely out of line with) C/L, and the advection term scales as:

$$u\frac{\partial c}{\partial x} \sim U\frac{C}{L}$$

Similarly, the second derivative $\partial^2 c/\partial x^2$ represents a difference of the gradient over a distance and is estimated at $(C/L)/\mathrm{L} = C/L^2$, and the diffusion term scales as:

$$D\frac{\partial^2 c}{\partial x^2} \sim D\frac{C}{L^2}$$

Equipped with these estimates, we can then compare the two processes by forming the ratio of their scales:

$$\frac{\text{advection}}{\text{diffusion}} = \frac{UC/L}{DC/L^2} = \frac{U\,L}{D}$$

This ratio is obviously dimensionless.
Traditionally, it is called the Peclet number and is denoted by Pe:

$$\boxed{Pe = \frac{U\,L}{D}}$$

Jean Claude Eugène Péclet
(1793 – 1857)

If $Pe << 1$ (in practice, if $Pe < 0.1$):
the advection term is significantly smaller than the diffusion term.

Physically, diffusion dominates and advection is negligible.

Spreading occurs almost symmetrically despite the directional bias of the (weak) flow.
If we wish to simplify the problem, we may drop the $u\partial c/\partial x$ term, as if u were nil.

The relative error committed in the solution by so doing is expected to be on the order of the Peclet number, and the smaller Pe, the smaller the error.

The solutions established with diffusion only were based on such simplification and are thus valid as long as $Pe << 1$.

If $Pe >> 1$ (in practice, if $Pe > 10$):
the advection term is significantly bigger than the diffusion term.

Physically, advection dominates and diffusion is negligible, and spreading is almost inexistent, with the patch of pollutant being simply moved along by the flow.

If we wish to simplify the problem, we may drop the $D\partial^2 c/\partial x^2$ term, as if D were nil.

The relative error committed in the solution by so doing is expected to be on the order of the inverse of the Peclet number ($1/Pe$), and the larger Pe, the smaller the error.

Note in this case:
that the neglect of the term with the highest-order derivative reduces the need of boundary conditions by one. No boundary condition may be imposed at the downstream end of the domain, and what happens there is whatever the flow brings.

The prototypical solution of the 1D advection only equation is:

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = 0 \quad \rightarrow \quad c(x,t) = c_0(x - ut)$$

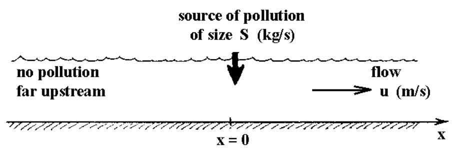
in which $c_0(x)$ is the initial concentration distribution.

If $Pe \sim 1$ (in practice, if $0.1 < Pe < 10$):

the advection and diffusion terms are not significantly different,
and neither process dominates over the other.

No approximation to the equation can be justified, and the full equation must be utilized.

Continuous discharge in a moving stream, with decay



The budget equation is

$$\cancel{\frac{\partial c}{\partial t}} + u\frac{\partial c}{\partial x} = D\frac{\partial^2 c}{\partial x^2} - Kc + \cancel{s} \qquad \text{with } u > 0$$

steady state

punctual source

We look for an exponential solution of the form $c(x) \div e^{\lambda x}$

and substitution in the budget equation yields:

$$D\lambda^2 - u\lambda - K = 0$$

This algebraic quadratic equation possesses two solutions:

$$\lambda_+ = \frac{u + \sqrt{u^2 + 4DK}}{2D}$$
$$\lambda_- = \frac{u - \sqrt{u^2 + 4DK}}{2D}$$

The first root is always positive while the second is always negative (since $u > 0$).
It follows that only one exponential is retained on each side of the discharge point:

$$c(x) = A\,e^{\lambda_+ x} \quad \text{for} \quad x < 0$$
$$c(x) = A\,e^{\lambda_- x} \quad \text{for} \quad x > 0$$

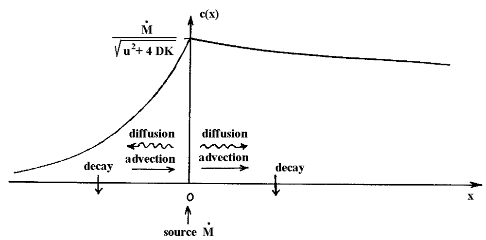
in which the constant A of integration is the same in each expression to ensure continuity of the concentration at $x = 0$.

The balance of fluxes in the vicinity of the source, which stipulates that what comes from the upstream side plus what comes from the source ought to be equal to what goes downstream, demands:

$$q(x=0-)+\dot{M}=q(x=0+) \quad \text{with} \quad q=cu-D\frac{dc}{dx}$$
$$\Rightarrow \quad (u-\lambda_+ D)\,A+\dot{M}=(u-\lambda_- D)\,A$$
$$\Rightarrow \quad A=\frac{\dot{M}}{(\lambda_+-\lambda_-)\,D}=\frac{\dot{M}}{\sqrt{u^2+4DK}}$$

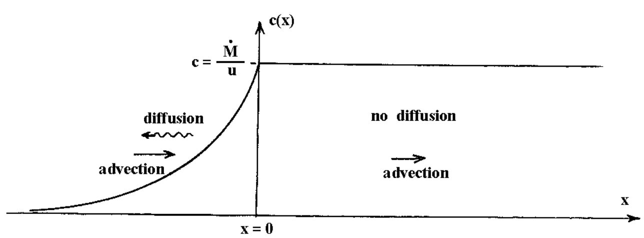
With the constant A now determined, we can write the solution:

$$c(x)=\frac{\dot{M}\,e^{\lambda_+ x}}{\sqrt{u^2+4DK}} \quad \text{for} \quad x\le 0$$
$$c(x)=\frac{\dot{M}\,e^{\lambda_- x}}{\sqrt{u^2+4DK}} \quad \text{for} \quad x\ge 0$$
$$\Rightarrow \quad c_{\max}=c(x=0)=\frac{\dot{M}}{\sqrt{u^2+4DK}}$$



In the absence of decay:

$$K=0 \quad \Rightarrow \quad \lambda_+=\frac{u}{D} \quad \text{and} \quad \lambda_-=0$$



$$c(x)=\frac{\dot{M}\,e^{\frac{ux}{D}}}{u} \quad \text{for} \quad x\le 0$$
$$c(x)=\frac{\dot{M}}{u}=\text{constant} \quad \text{for} \quad x\ge 0$$
$$\Rightarrow \quad c_{\max}=c(x=0)=\frac{\dot{M}}{u}$$

Note dilution by the flow:
The faster the flow, the greater the u value, and the lower the downstream concentration.

Highly Advective Situations

Consider a 2D situation in which there is advection (direction taken as the x-axis) and diffusion in both downstream and transverse directions. The budget equation is:

$$\frac{\partial c}{\partial t} + u\,\frac{\partial c}{\partial x} = D\left(\cancel{\frac{\partial^2 c}{\partial x^2}} + \frac{\partial^2 c}{\partial y^2}\right)$$

Then assume that advection dominates over diffusion (high Peclet number).
In this case, $u\partial c/\partial x$ dominates over $D\partial^2 c/\partial x^2$.
Note that we need to retain the transverse diffusion $D\partial^2 c/\partial y^2$ term since this is the only transport mechanism in that direction.

If we may further assume steady state (dc/dt = 0), then the budget equation reduces to:

$$u\,\frac{\partial c}{\partial x} = D\,\frac{\partial^2 c}{\partial y^2}$$

which is isomorphic to the 1D diffusion-only equation

$$\frac{\partial c}{\partial t} = D\,\frac{\partial^2 c}{\partial x^2}$$

by substituting $x \rightarrow ut$ and $y \rightarrow x$.

By performing the same substitution in the 1D-diffusion solution,
we obtain the solution in the case of steady state advection with transverse diffusion:

$$\left.\begin{array}{l} c(x,t) = \dfrac{M}{\sqrt{4\pi\,Dt}}\exp\left(-\dfrac{x^2}{4Dt}\right) \\ x \rightarrow y \quad \text{and} \quad t \rightarrow \dfrac{x}{u} \end{array}\right\} \longrightarrow \quad c(x,y) = M\sqrt{\frac{u}{4\pi\,Dx}}\exp\left(-\frac{uy^2}{4Dx}\right)$$

The quantity M is to be interpreted as the amount of contaminant released per unit height and unit downstream direction (the “missing” dimensions z and x since diffusion operates in the y-direction).

But here the problem ought to be posed by specifying a source S per unit height and per time. The connection between the two is:

$$S = \frac{\text{amount released}}{\text{height} \times \text{time}} = \frac{\text{amount released}}{\text{height} \times \text{downstream length}} \times \frac{\text{downstream length}}{\text{time}} = M\ u$$

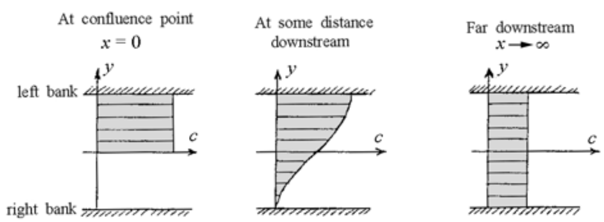
With $S = Mu$, the solution becomes:

$$c(x,y) = \frac{S}{\sqrt{4\pi\, Dux}} \exp\left(-\frac{uy^2}{4Dx}\right)$$

Width of spread is given by the 4σ-rule:

$$\text{Width} = 4\sigma = 5.66\sqrt{Dt} = 5.66\sqrt{\frac{Dx}{u}}$$

Example: Confluence of two streams, one of them polluted, the other clean



$$c(x,y) = \frac{c_0}{2}\left[1 + erf\left(y\sqrt{\frac{u}{4Dx}}\right)\right]$$

Example of transverse diffusion in a highly advective situation:

The mixing of the Rio Negro (black water)
with the Rio Amazonas (brown water)
in the Amazon jungle.