Environmental Transport and Fate

Chapter 2 – Diffusion – Part 3: With source and decay

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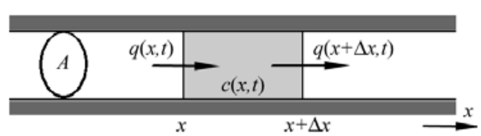
Adding source and decay in the diffusion equation

We now extends our analysis to include cases when the contaminant is not only diffusing but also replenished and decaying over time.

Recalling the earlier mass budget and applying it to an infinitesimal control volume of length Δx and cross-section A, we determine the import and export fluxes:



$$q_{in}\ A_{in} = q(x,t)\ A$$
$$q_{out}\ A_{out} = q(x+\Delta x,t)\ A$$

and the budget becomes

new terms

$$V\ \frac{dc}{dt} = +q(x,t)\ A - q(x+\Delta x,t)\ A + S - K\ V\ c$$

Since the volume of the small control volume is $V = A\Delta x$,

$$\frac{dc}{dt} = -\frac{q(x+\Delta x,t) - q(x,t)}{\Delta x} + \frac{S}{V} - K\ c$$

If we note $s = S\,/\,V$, the source per volume (*i.e.*, source density) and take the limit of a vanishing Δx, we get the differential equation

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} + s - Kc$$

Storage Diffusion Source Decay

which is the same as before, except for the two new terms.

The solution corresponding to an instantaneous (at t = 0) and localized (at x = 0) release with decay and no continuous source aside the initial release of amount M is:

$$c(x,t) = \frac{M}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt} - K\,t\right)$$

new term

Continuous release at a fixed location

As an application, consider the case of a continuous release at a fixed location, as is the case of a point source. Because the time-dependent problem is rather difficult to solve and also because the practical question in such situation may be limited to finding the ultimate state, we shall consider here only the **steady-state solution** to the problem (by putting $\partial c/\partial c$ to zero). Since the source is punctual (say, at $x = 0$), there is no source anywhere else ($s = 0$), and the governing equation reduces to:

$$0 = D\frac{\partial^2 c}{\partial x^2} - K\,c$$

The solution consists of two exponential functions:

$$c(x) = Ae^{+\lambda x} + Be^{-\lambda x}$$

in which A and B are two constants of integration to be determined by application of two boundary conditions, and the exponent λ is found by substituting the exponential solution in the original differential equation. We find:

$$\lambda = \sqrt{\frac{K}{D}}$$

For the solution, we need to separate the domain on the left of the point source from the portion of the domain on the right side of the source. Applying the condition $c \rightarrow 0$ as $x \rightarrow \pm\infty$, we knock down the growing exponentials, leaving:

$$c(x) = Ae^{+\lambda x} \quad \text{for} \quad -\infty < x < 0$$
$$c(x) = Be^{-\lambda x} \quad \text{for} \quad 0 < x < +\infty$$

Uniqueness of the concentration value at $x = 0$ demands continuity of $c(0)$ and thus

$$A = B.$$

This leaves a single unknown, the coefficient A.

This coefficient A is related to the amount being released. For an amount $\dot{M}$ (mass released per cross-sectional area of the domain and per time, in kg/m^2.s), half of each by symmetry goes to each side, and thus:

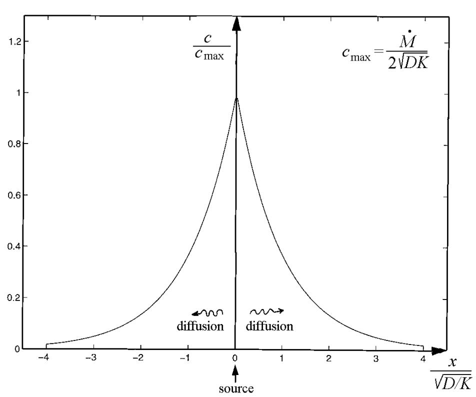
$$-D\frac{\partial c}{\partial x} = \frac{\dot{M}}{2} \quad \text{at} \quad x = 0 \quad \text{from either side}$$

From which follows:

$$A = B = \frac{\dot{M}}{2D\lambda} = \frac{\dot{M}}{2\sqrt{DK}}$$

Putting the pieces together, we arrive at the solution:

$$c(x) = \frac{\dot{M}}{2\sqrt{DK}} \exp\left(+\sqrt{\frac{K}{D}}\, x\right) \quad \text{for} \quad -\infty < x < 0$$
$$c(x) = \frac{\dot{M}}{2\sqrt{DK}} \exp\left(-\sqrt{\frac{K}{D}}\, x\right) \quad \text{for} \quad 0 < x < +\infty$$



Of interest is the maximum concentration:

$$c_{\max} = c(x = 0) = \frac{\dot{M}}{2\sqrt{DK}}$$

If spread is defined as the length of the region that contains 95% of the contamination, we have:

$$\text{Width} = \frac{6.0}{\lambda} = 6.0\sqrt{\frac{D}{K}}$$

Example application to the Chicago Ship Canal

We again return to our example with the Chicago Ship Canal and now take into account that benzene in water is subject to bacterial decay (aerobic degradation) at the known rate of K = 0.11/day. If a stationary barge containing benzene and parked along the side of the canal has been leaking over the last few weeks at the estimated rate of 2.5 liters per day, what is the benzene concentration in the canal water near the barge, and how far along the canal is the concentration in excess of the drinking-water standard of 0.005 mg/L?

We assume steady state and uniformity of the benzene concentration across the canal and in the vertical. (Recall: canal width= 48.8 m, depth= 8.07 m, along-canal diffusivity is D = 3.0 m^2/s, and benzene density = 0.879 g/cm^3.)

To solve this problem, we first need to determine $\dot{M}$, the rate of input per cross-area:

$$\dot{M} = \frac{(2.5\,\text{L/day})(879\,\text{g/L})}{(48.8\,\text{m})(8.07\,\text{m})(86{,}400\,\text{s/day})} = 6.458\times10^{-5}\ \frac{\text{g}}{\text{m}^2\cdot\text{s}}$$

We also rewrite the decay constant using seconds as the time unit:

$$K = \frac{0.11/\text{day}}{86{,}400\,\text{s/day}} = 1.273\times10^{-6}\ /\text{s}$$

From these elements, we can determine the benzene concentration in the canal section at the position of the leaky barge:

$$c_{\max} = \frac{\dot{M}}{2\sqrt{DK}} = 0.0165\,\text{g/m}^3 = 0.0165\,\text{mg/L}$$

This concentration exceeds the drinking standard c_{std} = 0.005 g/m^3.
We can find the distance x where the concentration falls to the drinking standard by inverting the solution:

$$x = -\sqrt{\frac{D}{K}}\ \ln\left(\frac{2\,c_{\text{std}}\sqrt{DK}}{\dot{M}}\right) = 1{,}835\,\text{m}$$

Thus, the benzene concentration exceeds the drinking standard in a zone of 3.67 kilometers along the canal, centered on the barge position.

Course notes follow with additional examples, for reference only and not presented in lectures.